Momentum Operator in Coordinate Space

Wave-particle duality is at the heart of quantum mechanics. A particle with wavelength 8 has wave function (un-normalized)

$$\langle x | \lambda \rangle = \exp\left(i 2\pi \frac{x}{\lambda} \right)$$

However, according to deBroglie's wave equation the particle's momentum is $p = h/$8. Therefore the momentum wave function of the particle in coordinate space is

$$\langle x | p \rangle = \exp\left(\frac{ipx}{\hbar} \right)$$

In momentum space the following eigenvalue equation holds: $\hat{p} | p \rangle = p | p \rangle$. Operating on the momentum eigenfunction with the momentum operator in momentum space returns the momentum eigenvalue times the original momentum eigenfunction. In other words, in its own space the momentum operator is a multiplicative operator (the same is true of the position operator in coordinate space). To obtain the momentum operator in coordinate space this expression can be projected onto coordinate space by operating on the left by $\langle x |$.

$$\langle x | \hat{p} | p \rangle = p \langle x | p \rangle = p \exp\left(\frac{ipx}{\hbar} \right) = \frac{\hbar}{i} \frac{d}{dx} \langle x | p \rangle$$

Comparing the first and last terms reveals that

$$\langle x | \hat{p} = \frac{\hbar}{i} \frac{d}{dx} \langle x |$$

and that $\frac{\hbar}{i} \frac{d}{dx}$ is the momentum operator in coordinate space.

The position wave function in momentum space is the complex conjugate of the momentum wave function coordinate space.

$$\langle p | x \rangle = \langle x | p \rangle^* = \exp\left(\frac{-ipx}{\hbar} \right)$$

Using the method outlined above it is easy to show that the position operator in momentum space is $-\frac{\hbar}{i} \frac{d}{dp}$.