Examining the Wigner Distribution Using Dirac Notation

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Abstract: Expressing the Wigner distribution function in Dirac notation reveals its resemblance to a classical trajectory in phase space.

References to the Wigner distribution function [1-3] and the phase-space formulation of quantum mechanics are becoming more frequent in the pedagogical and review literature [4-26]. There have also been several important applications reported in the recent research literature [27, 28]. Other applications of the Wigner distribution are cited in Ref. 25.

The purpose of this note is to demonstrate that when expressed in Dirac notation the Wigner distribution resembles a classical phase-space trajectory. The Wigner distribution can be generated from either the coordinate- or momentum-space wave function. The coordinate-space wave function will be employed here and the Wigner transform using it is given in equation (1) for a one-dimensional example in atomic units.

$$W(p,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi^*(x+\tfrac{s}{2})\Psi(x-\tfrac{s}{2})e^{ips}ds \qquad \textbf{(1)}$$

In Dirac notation the first two terms of the integrand are written as follows,

$$\Psi^*(x+\tfrac{s}{2}) = \left\langle \Psi \middle| x+\tfrac{s}{2}\right\rangle \qquad \Psi(x-\tfrac{s}{2}) = \left\langle x-\tfrac{s}{2} \middle| \Psi\right\rangle \qquad \textbf{(2)}$$

Assigning 1/2π to the third term and utilizing the momentum eigenfunction in coordinate space and its complex conjugate we have,

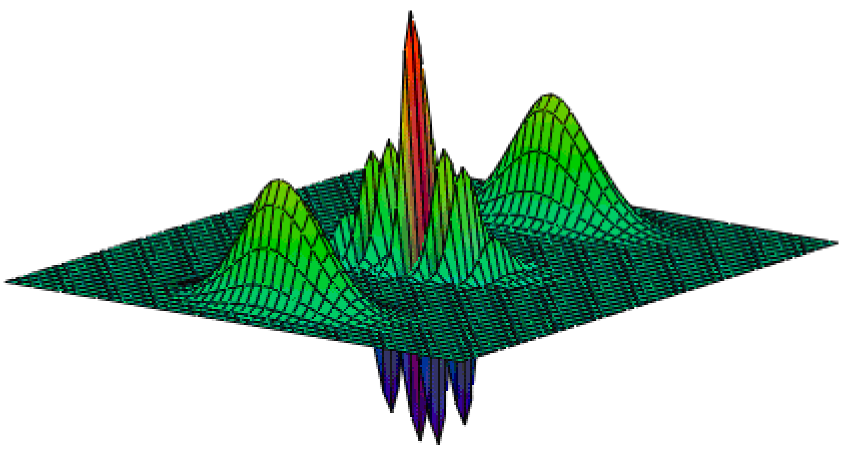
$$\tfrac{1}{2\pi}e^{ips} = \frac{1}{\sqrt{2\pi}}e^{ip(x+\frac{s}{2})}\frac{1}{\sqrt{2\pi}}e^{-ip(x-\frac{s}{2})} = \left\langle x+\tfrac{s}{2} \middle| p\right\rangle\left\langle p \middle| x-\tfrac{s}{2}\right\rangle \qquad \textbf{(3)}$$

Substituting equations (2) and (3) into equation (1) yields after arrangement,

$$W(x,p)=\int_{-\infty}^{+\infty}\left\langle\Psi\middle|x+\tfrac{s}{2}\right\rangle\left\langle x+\tfrac{s}{2}\middle|p\right\rangle\left\langle p\middle|x-\tfrac{s}{2}\right\rangle\left\langle x-\tfrac{s}{2}\middle|\Psi\right\rangle ds \qquad \textbf{(4)}$$

The four Dirac brackets are read from right to left as follows: (1) is the amplitude that a particle in the state Ψ has position (*x* - *s/2*); (2) is the amplitude that a particle with position (*x* - *s/2*) has momentum *p*; (3) is the amplitude that a particle with momentum *p* has position (*x* + *s/2*); (4) is the amplitude that a particle with position (*x* + *s/2*) is (still) in the state Ψ. Thus, in Dirac notation the integrand is the quantum equivalent of a classical phase-space trajectory for a quantum system in the state Ψ.

Integration over *s* creates a superposition of all possible quantum trajectories of the state Ψ, which interfere constructively and destructively, providing a quasi-probability distribution in phase space. As an example, the Wigner probability distribution for a double-slit experiment is shown in the figure below [14, 27]. The oscillating positive and negative values in the middle of the Wigner distribution signify the interference associated with a quantum superposition, distinguishing it from a classical phase-space probability distribution. In the words of Leibfried et al. [14], the Wigner function is a “mathematical construct for visualizing quantum trajectories in phase space.”



Wigner distribution function for the double-slit experiment.

The Wigner double- and triple-slit distribution functions are calculated in the following tutorials.

http://www.users.csbsju.edu/~frioux/wigner/Wigner-2-Slit.pdf

http://www.users.csbsju.edu/~frioux/wigner/Wigner-3-Slit.pdf

Examples of the generation and use of the Wigner distribution are available via the following links.

http://www.users.csbsju.edu/~frioux/wigner/wigner-pib.pdf

http://www.users.csbsju.edu/~frioux/wigner/WignerSHO.pdf

http://www.users.csbsju.edu/~frioux/wigner/Wigner1DHatom-rev.pdf

http://www.users.csbsju.edu/~frioux/VariMethod/WignerFeshbachVari.pdf

Phase-space quantum mechanical calculations using the Wigner distribution are compared with coordinate- and momentum-space calculations in the following tutorial.

http://www.users.csbsju.edu/~frioux/wigner/CoordMomPhase.pdf

The Cliff Notes version of the above can be accessed via the following link.

http://www.users.csbsju.edu/~frioux/wigner/PhaseSpaceQM.pdf

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