Wigner Distribution for the Particle in a Box

The Wigner function is a quantum mechanical phase-space quasi-probability function. It is called a quasi-probability function because it can take on negative values, which have no classical meaning in terms of probability.

The PIB eigenstates for a box of unit dimension are given by $\Psi(x,n) := \sqrt{2}\cdot\sin(n\cdot\pi\cdot x)$

For these eigenstates the Wigner distribution function is:

$$W(x,p,n) := \frac{1}{\pi}\cdot\int_{-x}^{x}\sqrt{2}\cdot\sin[n\cdot\pi\cdot(x+s)]\cdot\exp(2\cdot i\cdot s\cdot p)\cdot\sqrt{2}\cdot\sin[n\cdot\pi\cdot(x-s)]\,ds$$

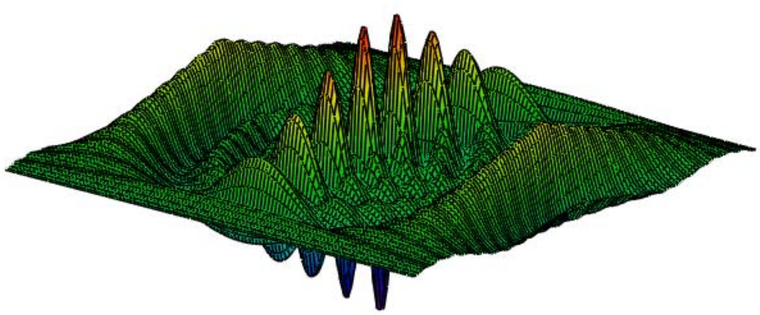
Integration with respect to s yields the following function:

$$W(x,p,n) := \frac{2}{\pi}\cdot\left[\frac{\sin[2\cdot(p-n\cdot\pi)\cdot x]}{4\cdot(p-n\cdot\pi)}+\frac{\sin[2\cdot(p+n\cdot\pi)\cdot x]}{4\cdot(p+n\cdot\pi)}-\cos(2\cdot n\cdot\pi\cdot x)\cdot\frac{\sin(2\cdot p\cdot x)}{2\cdot p}\right]$$

The Wigner distribution for the nth eigenstate is calculated below: $n := 10$

$N := 115$ $\quad i := 0..N$ $\quad x_i := \frac{i}{N}$ $\quad j := 0..N$ $\quad p_j := -40+\frac{80\cdot j}{N}$

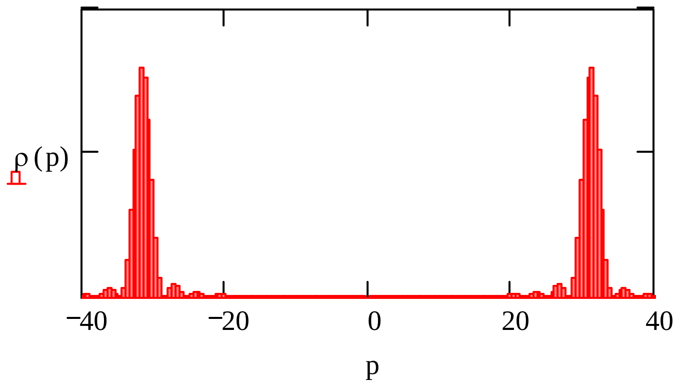
$$Wigner_{i,j} := \text{if}\left[x_i \le .5, W(x_i,p_j,n), W[(1-x_i),p_j,n]\right]$$



Wigner

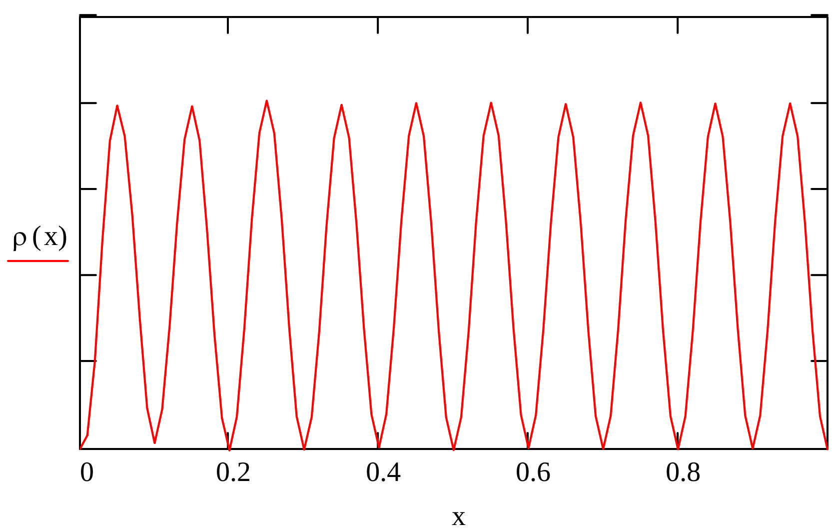
Integration of the Wigner function over the spatial coordinate yields the momentum distribution function as is shown below.

$$\rho(p) := \int_0^1 W(x,p,n)\,dx \qquad p := -40,-39.5..40$$



Integration of the Wigner function over the momentum coordinate yields the spatial distribution function as is shown below.

$$\rho(x) := \int_{-51}^{50} W(x,p,n)\,dp \qquad x := 0,.01..1$$



The Wigner distribution can be used to calculate the expectation values for position, momentum and kinetic energy.

$$x_{bar} = \int_{-\infty}^{\infty}\int_{0}^{1} W(x,p,1)\cdot x \, dx \, dp \text{ simplify } \rightarrow x_{bar} = \frac{1}{2}$$

$$p_{bar} = \int_{-\infty}^{\infty}\int_{0}^{1} W(x,p,1)\cdot p \, dx \, dp \text{ simplify } \rightarrow p_{bar} = 0$$

$$T_{bar} = \int_{-\infty}^{\infty}\int_{0}^{1} W(x,p,1)\cdot \frac{p^2}{2} \, dx \, dp \text{ simplify } \rightarrow T_{bar} = \frac{1}{2}\cdot\pi^2$$

References:

"Wigner quasi-probability distribution for the infinite square well: Energy eigenstates and time-dependent wave packets," by Belloni, Docheski and Robinett; *American Journal of Physics* **72(9)**, 1183-1192 (2004).

"Wigner functions and Weyl transforms for pedestrians," by William Case, *American Journal of Physics* **76(10)**, 937-946 (2008).