A Stern-Gerlach Quantum "Eraser"

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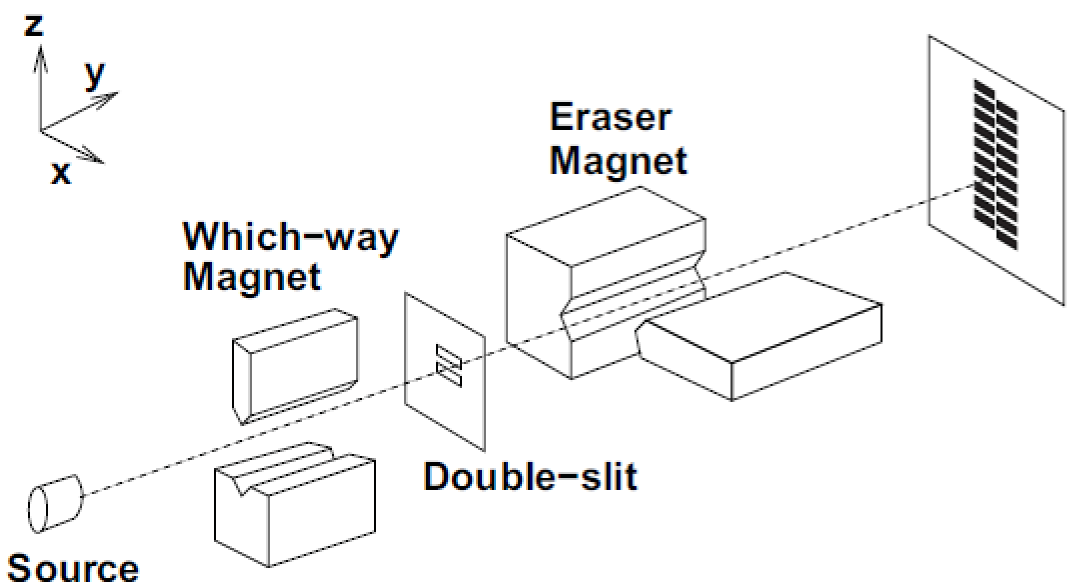
This tutorial provides a brief mathematical analysis of a proposed quantum eraser experiment involving spin-1/2 particles which is available at arXiv:quant-ph/0501010v2. Please see the two immediately preceding tutorials for another example of the quantum eraser and additional mathematical detail.



The first magnet attaches which-way information such that the spin-1/2 particles leaving the double-slit screen are described by the following entangled wave function,

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_z\right\rangle\left|z_1\right\rangle + \left|\downarrow_z\right\rangle\left|z_2\right\rangle\right]$$

where z_1 and z_2 represent the positions of the horizontal slits on the z-axis and the spin eigenstates in the z-direction are given below.

$$\Psi_{zup} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Psi_{zdown} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Recognizing that a diffraction pattern is actually a momentum distribution function, we project Ψ onto momentum space as follows (in atomic units, $h = 2\pi$).

$$\left\langle p|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_z\right\rangle\left\langle p|z_1\right\rangle + \left|\downarrow_z\right\rangle\left\langle p|z_2\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\left|\uparrow_z\right\rangle\frac{\exp(-ipz_1)}{\sqrt{2\pi}} + \left|\downarrow_z\right\rangle\frac{\exp(-ipz_2)}{\sqrt{2\pi}}\right]$$

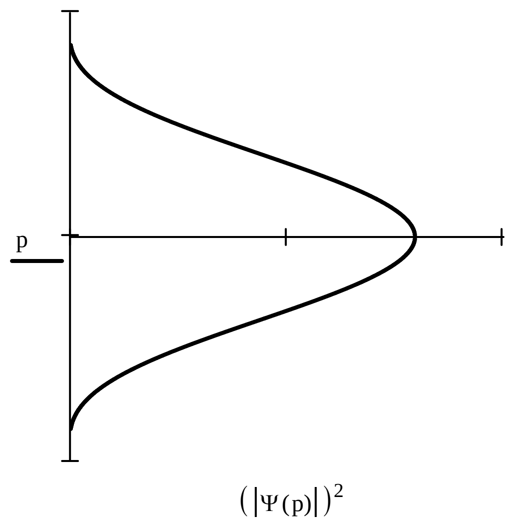
Here the exponential terms are the position eigenfunctions in momentum space for infinetesimally thin slits located at z_1 and z_2. For slits of finite width $\langle p|\Psi\rangle$ is written as shown below. Again see the previous tutorials in this series for further mathematical detail. The slit positions and slit width chosen are arbitrary.

Slit positions: $z_1 := 1$ $z_2 := 2$ Slit width: $\delta := .2$

$$\Psi(p) := \frac{1}{\sqrt{2}} \cdot \left(\Psi_{zup} \cdot \int_{z_1 - \frac{\delta}{2}}^{z_1 + \frac{\delta}{2}} \frac{1}{\sqrt{2 \cdot \pi}} \cdot \exp(-i \cdot p \cdot z) \cdot \frac{1}{\sqrt{\delta}} \, dz + \Psi_{zdown} \cdot \int_{z_2 - \frac{\delta}{2}}^{z_2 + \frac{\delta}{2}} \frac{1}{\sqrt{2 \cdot \pi}} \cdot \exp(-i \cdot p \cdot z) \cdot \frac{1}{\sqrt{\delta}} \, dz \right)$$

Because of the addition of path information there are no interference fringes associated with this two-slit wave function; the encoded orthogonal z-direction eigenstates destroy the interference cross terms as shown graphically below.

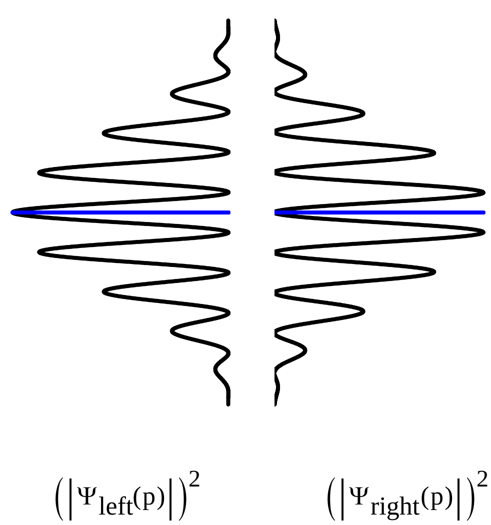
$$p := -30, -29.98 .. 30$$



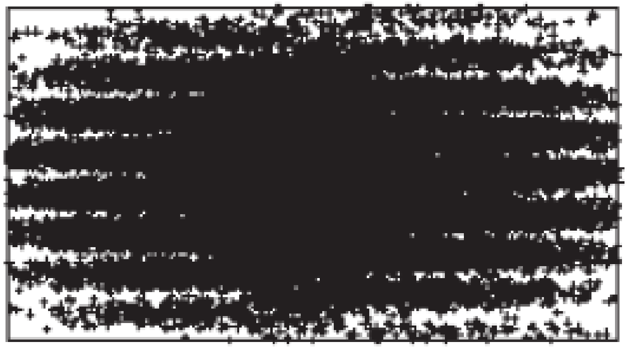
The second Stern-Gerlach magnet oriented in the x-direction, according to the conventional interpretation, "erases" the which-way information. This is shown by projecting the state after the first magnet and the slit screen, Ψ(p), onto the x-direction spin eigenstates.

$$\Psi_{xup} := \frac{1}{\sqrt{2}} \cdot (1 \quad 1) \qquad \Psi_{xdown} := \frac{1}{\sqrt{2}} \cdot (1 \quad -1)$$

$$\Psi_{left}(p) := \Psi_{xup} \cdot \Psi(p) \qquad \Psi_{right}(p) := \Psi_{xdown} \cdot \Psi(p)$$



The two halves of this figure capture the basic features of Figure 2b in the reference cited and shown below.

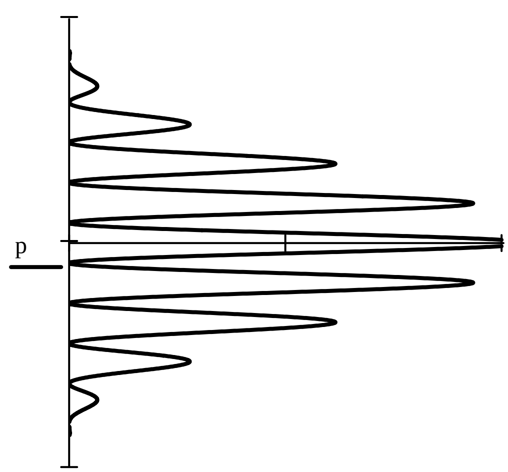


The horizontal blue line marks p = 0 on the z-axis. On the left is the interference pattern of the part of the beam emerging from the x-up magnet direction with spin state Ψ_{xup}, and on the right is the interference pattern of the part of the beam emerging from the x-down magnet direction with spin state Ψ_{xdown}. As shown in the Summary, Ψ(p) can be rewritten in terms of the x-direction spin states clearly showin the superpositions responsible for the interference fringes on the left and right.

$$\langle p|\Psi\rangle = \frac{1}{2}\left[|\uparrow_x\rangle\left(\langle p|z_1\rangle + \langle p|z_2\rangle\right) + |\downarrow_x\rangle\left(\langle p|z_1\rangle - \langle p|z_2\rangle\right)\right]$$

In the absence of both Stern-Gerlach magnets the usual double-slit interference pattern is observed.

$$\Psi(p) := \frac{1}{\sqrt{2}}\cdot\left(\int_{z_1-\frac{\delta}{2}}^{z_1+\frac{\delta}{2}} \frac{1}{\sqrt{2\cdot\pi}}\cdot\exp(-i\cdot p\cdot z)\cdot\frac{1}{\sqrt{\delta}}\,dz + \int_{z_2-\frac{\delta}{2}}^{z_2+\frac{\delta}{2}} \frac{1}{\sqrt{2\cdot\pi}}\cdot\exp(-i\cdot p\cdot z)\cdot\frac{1}{\sqrt{\delta}}\,dz\right)$$



Alternative Analysis

It is possible to express the mathematics in an alternative but equivalent form. The first wave function,

$$\frac{1}{\sqrt{2}}\left[\left|\uparrow_z\right\rangle\left|z_1\right\rangle+\left|\downarrow_z\right\rangle\left|z_2\right\rangle\right] \qquad \text{A}$$

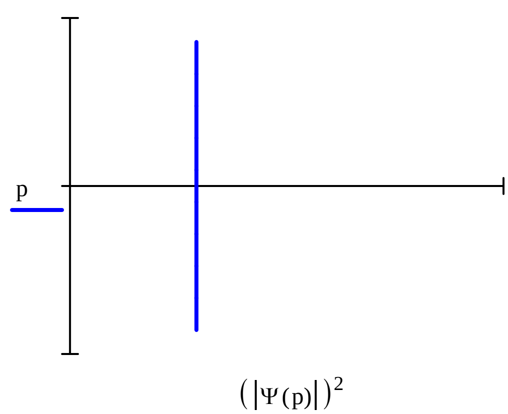
can be expressed explicitly in vector format in the momentum representation. This analysis will be based on infinitesimally thin slits as introduced earlier.

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\left|z_1\right\rangle+\begin{pmatrix}0\\1\end{pmatrix}\left|z_2\right\rangle\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}\left|z_1\right\rangle\\\left|z_2\right\rangle\end{pmatrix}\xrightarrow{\langle p|}\frac{1}{\sqrt{2}}\begin{pmatrix}\left\langle p|z_1\right\rangle\\\left\langle p|z_2\right\rangle\end{pmatrix} \qquad \text{B}$$

It is easily shown that this wave function does not lead to interference fringes at the detection screen by calculating the square of its absolute magnitude.

$$\frac{1}{2}\begin{pmatrix}\left\langle z_1|p\right\rangle & \left\langle z_2|p\right\rangle\end{pmatrix}\begin{pmatrix}\left\langle p|z_1\right\rangle\\\left\langle p|z_2\right\rangle\end{pmatrix}=\frac{1}{2}\left[\left|\left\langle p|z_1\right\rangle\right|^2+\left|\left\langle p|z_2\right\rangle\right|^2\right] \qquad \text{C}$$

$$\Psi(p) := \frac{1}{2\cdot\sqrt{\pi}}\cdot\begin{pmatrix}\exp(-i\cdot p\cdot z_1)\\\exp(-i\cdot p\cdot z_2)\end{pmatrix}$$



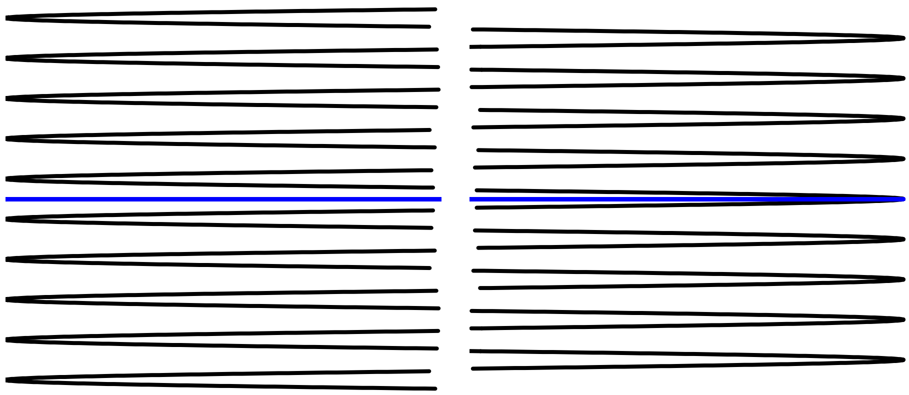
However, in the presence of the second Stern-Gerlach magnet vector B is projected onto the two output channels of the magnet.

$$\frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\left\langle p|z_1\right\rangle\\\left\langle p|z_2\right\rangle\end{pmatrix}=\frac{1}{2}\left[\left\langle p|z_1\right\rangle+\left\langle p|z_2\right\rangle\right]$$

$$\frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}\left\langle p|z_1\right\rangle\\\left\langle p|z_2\right\rangle\end{pmatrix}=\frac{1}{2}\left[\left\langle p|z_1\right\rangle-\left\langle p|z_2\right\rangle\right]$$

The probability distributions of these states show interference fringes.

$$\Psi_{left}(p) := \frac{1}{\sqrt{2}} \cdot (1 \;\; 1) \cdot \frac{1}{2 \cdot \sqrt{\pi}} \cdot \begin{pmatrix} \exp(-i \cdot p \cdot z_1) \\ \exp(-i \cdot p \cdot z_2) \end{pmatrix} \qquad \Psi_{right}(p) := \frac{1}{\sqrt{2}} \cdot (1 \;\; -1) \cdot \frac{1}{2 \cdot \sqrt{\pi}} \cdot \begin{pmatrix} \exp(-i \cdot p \cdot z_1) \\ \exp(-i \cdot p \cdot z_2) \end{pmatrix}$$



$$\left(\left|\Psi_{left}(p)\right|\right)^2 \qquad \left(\left|\Psi_{right}(p)\right|\right)^2$$

Summary

The z-direction Stern-Gerlach magnet and the slit screen create the following entangled superposition which does not produce interference fringes due to the orthogonality of the spin states marking the slits.

$$\langle p|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_z\rangle\langle p|z_1\rangle + |\downarrow_z\rangle\langle p|z_2\rangle\right]$$

To understand what happens at the x-direction magnet this state is rewritten in the x-direction spin basis.

$$\langle p|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle + |\downarrow_x\rangle\right)\langle p|z_1\rangle + \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle - |\downarrow_x\rangle\right)\langle p|z_2\rangle\right]$$

Collecting terms on the x-direction spin eigenstates yields,

$$\langle p|\Psi\rangle = \frac{1}{2}\left[|\uparrow_x\rangle\left(\langle p|z_1\rangle + \langle p|z_2\rangle\right) + |\downarrow_x\rangle\left(\langle p|z_1\rangle - \langle p|z_2\rangle\right)\right]$$

The in-phase and out-of-phase superpositions, highlighted in blue and red, exit the magnet in opposite directions. Because of this the superpositions become spatially separated which leads to two sets of interference fringes with a one-fringe relative phase shift at the detection screen.

It's clear to me that erasure is not a satisfactory explanation for this process. Because fringes appear after the x-direction magnet it might seem plausible, at first glance, to assume that the which-way markers have been erased. But actually the x-direction magnet sorts $< p \mid \Psi >$ into two components in terms of the x-direction spin eigenstates. Nothing has been erased.