Density Operator Approach to the Double-Slit Experiment

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A sharply focused particle beam (photons, electrons, molecules, etc.) is incident on a screen with two slits. According to quantum mechanics the individual particles are represented by a coherent superposition of being simultaneously at both slits.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle + |2\rangle\right]$$

In the interest of mathematical simplicity, *1* and *2* label slits that are infinitesimally narrow in the x-direction and infinitely long in the y-direction.

The density operator for this state is

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left[|1\rangle + |2\rangle\right]\left[\langle 1| + \langle 2|\right] = \frac{1}{2}\left[|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|\right]$$

The expectation value for the arrival of a particle at position *x* on the detection screen is

$$\langle x|\hat{\rho}|x\rangle = \frac{1}{2}\left[\langle x|1\rangle\langle 1|x\rangle + \langle x|1\rangle\langle 2|x\rangle + \langle x|2\rangle\langle 1|x\rangle + \langle x|2\rangle\langle 2|x\rangle\right]$$

Rearrangement yields,

$$\langle x|\hat{\rho}|x\rangle = \frac{1}{2}\left[\left|\langle x|1\rangle\right|^2 + \left|\langle x|2\rangle\right|^2 + \langle x|1\rangle\langle x|2\rangle^* + \langle x|2\rangle\langle x|1\rangle^*\right]$$

The probability amplitudes in this equation represent the phase of a particle on arrival at position *x* from slits *1* and *2*. For example, using Euler's equation we calculate the phase of a particle arriving at x from slit *1* as follows,

$$\langle x|1\rangle = \frac{1}{\sqrt{2\pi}}\exp\left(i2\pi\frac{\delta x_1}{\lambda}\right)$$

where δx_1 is the distance from slit *1* to position *x* on the detection screen and λ is the de Broglie wavelength of the particle.

Using this form for the probability amplitudes we can write the expectation value in terms of the distances to *x* from slits *1* and *2*.

$$\langle x|\hat{\rho}|x\rangle = \frac{1}{4\pi}\left[2+\exp\left(i2\pi\frac{(\delta x_1-\delta x_2)}{\lambda}\right)+\exp\left(-i2\pi\frac{(\delta x_1-\delta x_2)}{\lambda}\right)\right]$$
$$= \frac{1}{2\pi}\left[1+\cos\left(2\pi\frac{(\delta x_1-\delta x_2)}{\lambda}\right)\right]$$

Clearly $\frac{\delta x_1-\delta x_2}{\lambda}$ will vary continuously along the x-axis of the detector from large negative values at one end to large positive values at the other end leading to minima and maxima in the cosine term and therefore $\langle x|\hat{\rho}|x\rangle$, thereby yielding the well-known interference fringes associated with the double-slit experiment. Naturally a more realistic slit geometry will lead to a mathematically more complicated expression for the expectation value.

If one takes a classical view of the double-slit experiment that assumes the particle goes through one slit or the other, and has a 50% chance of going through either slit, the coherent superposition, $|\Psi\rangle$, is no longer a valid representation of the experiment. Under the classical view a density operator must be used, and in this case is

$$\hat{\rho}_{cl} = \tfrac{1}{2}|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|$$

The expectation value for the arrival of the particle at x on the detection screen is now,

$$\langle x|\hat{\rho}_{cl}|x\rangle = \tfrac{1}{2}\langle x|1\rangle\langle 1|x\rangle + \tfrac{1}{2}\langle x|2\rangle\langle 2|x\rangle = \tfrac{1}{2}|\langle x|1\rangle|^2 + \tfrac{1}{2}|\langle x|2\rangle|^2 = \text{constant}$$

which has a constant value with no oscillations in arrival probability as a function of *x*. In other words, no interference fringes.