

Addition of Spin Angular Momentum: A Tensor Algebra Analysis

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In this entry I work through section 4.4.3 of David Griffiths' *Introduction to Quantum Mechanics* (2nd ed.) in which he treats the addition of angular momentum for two identical spin-1/2 particles. The tensor algebra approach is illustrated.

The four spin states of two spin-1/2 particles are written below in the spin-z basis in tensor format.

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\uparrow\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The two middle states are not permissible because they distinguish between identical particles. The solution is to form symmetric and anti-symmetric superpositions of them, but let's follow Griffith's approach for the time being. The four spin states are labeled as shown below.

$$\mathbf{a} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{b} := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{c} := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{d} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The identity and spin operators in units of $\hbar/2\pi$ are now defined. The final two are the up and down spin ladder operators.

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{S}_x := \frac{1}{2} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{S}_y := \frac{1}{2} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathbf{S}_z := \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{S} := \mathbf{S}_x + \mathbf{S}_y + \mathbf{S}_z$$

$$\mathbf{S}_u := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathbf{S}_d := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Next the total spin operator and total spin operator in the z-direction are defined using **kronecker**, Mathcad's command for tensor matrix multiplication.

$$\mathbf{S}_{\text{tot}} := \text{kronecker}(\mathbf{S}, \mathbf{I}) + \text{kronecker}(\mathbf{I}, \mathbf{S}) \quad \mathbf{S}_{z_{\text{tot}}} := \text{kronecker}(\mathbf{S}_z, \mathbf{I}) + \text{kronecker}(\mathbf{I}, \mathbf{S}_z)$$

Calculation of the expectation values for total spin in the z-direction for spin states **a**, **b**, **c** and **d** reveals the problem mentioned above. For $S = 1$ we expect S_z values of -1, 0 and 1. The extra value for $S_z = 0$ indicates an interpretive problem.

$$\mathbf{a}^T \cdot \mathbf{S}_{z_{\text{tot}}} \cdot \mathbf{a} = 1 \quad \mathbf{b}^T \cdot \mathbf{S}_{z_{\text{tot}}} \cdot \mathbf{b} = 0 \quad \mathbf{c}^T \cdot \mathbf{S}_{z_{\text{tot}}} \cdot \mathbf{c} = 0 \quad \mathbf{d}^T \cdot \mathbf{S}_{z_{\text{tot}}} \cdot \mathbf{d} = -1$$

Griffiths solves the problem by operating with the lowering operator on spin state **a**, which yields an in-phase superposition (unnormalized) of spin states **b** and **c**.

$$\left(\text{kroncker}(S_d, I) + \text{kroncker}(I, S_d)\right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Repeating with the result from above yields an unnormalized **d** spin state.

$$\left(\text{kroncker}(S_d, I) + \text{kroncker}(I, S_d)\right) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Operating on the unnormalized **d** spin state yield a null vector suggesting that the original spin states might be reformulated as triplet and singlet states.

$$\left(\text{kroncker}(S_d, I) + \text{kroncker}(I, S_d)\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Given this hint and the fact that the initial spin states are orthonormal, we preserve this property in the new spin states by constructing an out-of-phase superposition of **b** and **c**. This give us a revised set of orthonormal spin vectors. In conventional notation these states are $|11\rangle$, $|10\rangle$, $|1-1\rangle$ and $|00\rangle$, where the first number is total spin value and the second the spin in the z-direction. Thus, we have a triplet and a singlet state as will be confirmed below.

$$\Psi_{1p1} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Psi_{10} := \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \Psi_{1m1} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \Psi_{00} := \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

We now establish by calculation of expectation values for S^2 (constructed below) and S_z that the first three spin states are members of a triplet state ($S = 1$) and the final spin state is a singlet ($S = 0$). The eigenvalues for the S^2 operator are $S(S + 1)$ which is 2 for the triplet state and 0 for the singlet state.

$$\begin{aligned} SS &:= \text{kroncker}(S^2, I) + \text{kroncker}(I, S^2) + 2 \cdot (\text{kroncker}(S_x, S_x) + \text{kroncker}(S_y, S_y) + \text{kroncker}(S_z, S_z)) \\ (\Psi_{1p1})^T \cdot SS \cdot \Psi_{1p1} &= 2 & (\Psi_{10})^T \cdot SS \cdot \Psi_{10} &= 2 & (\Psi_{1m1})^T \cdot SS \cdot \Psi_{1m1} &= 2 & (\Psi_{00})^T \cdot SS \cdot \Psi_{00} &= 0 \\ (\Psi_{1p1})^T \cdot Sz_{tot} \cdot \Psi_{1p1} &= 1 & (\Psi_{10})^T \cdot Sz_{tot} \cdot \Psi_{10} &= 0 & (\Psi_{1m1})^T \cdot Sz_{tot} \cdot \Psi_{1m1} &= -1 & (\Psi_{00})^T \cdot Sz_{tot} \cdot \Psi_{00} &= 0 \end{aligned}$$

Next, we look at this problem from the energy perspective and use the spin-spin interaction Hamiltonian to calculate the energy eigenvalues and eigenvectors for the two spin-1/2 particle system.

$$H_{\text{SpinSpin}} := \text{kroncker}(S_x, S_x) + \text{kroncker}(S_y, S_y) + \text{kroncker}(S_z, S_z)$$

$$H_{\text{SpinSpin}} = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0.5 & 0 \\ 0 & 0.5 & -0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{pmatrix}$$

We now ask Mathcad to calculate the eigenvalues and eigenvectors of the spin-spin operator. These results are displayed by constructing a matrix which contains the eigenvalues in the top row, and their eigenvectors in the columns below the eigenvalues.

$$i := 1..4 \quad E := \text{eigenvals}(H_{\text{SpinSpin}}) \quad \text{EigenvalEigenvec} := \text{rsort}\left(\text{stack}\left(E^T, \text{eigenvecs}(H_{\text{SpinSpin}})\right), 1\right)$$

$$\text{EigenvalEigenvec} = \begin{pmatrix} -0.75 & 0.25 & 0.25 & 0.25 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0 & 0.707 \\ -0.707 & 0 & 0 & 0.707 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

These calculations are consistent with those that preceded and with the results presented on page 284 of Griffiths' text.