Positronium Annihilation

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The following provides an alternative mathematical analysis of the annihilation of positronium as presented in section 18-3 of Volume III of *The Feynman Lectures on Physics*. Positronium is an analog of the hydrogen atom in which the proton is replaced by a positron, the electron's anti-particle. The electron-positron pair undergoes annihilation in 10^{-10} seconds producing two γ-ray photons.

Feynman shows that when positronium annihilates, conservation of momentum requires that the photons emitted in opposite directions (A and B) must have the same circular polarization state, either both in the right or both in left circular state in their direction of motion. This leads to the following entangled superposition in the R-L basis. The negative sign is required by parity conservation.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|R\rangle_A|R\rangle_B - |L\rangle_A|L\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\end{pmatrix}_A \otimes \begin{pmatrix}1\\i\end{pmatrix}_B - \begin{pmatrix}1\\-i\end{pmatrix}_A \otimes \begin{pmatrix}1\\-i\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\\i\\-1\end{pmatrix} - \begin{pmatrix}1\\-i\\-i\\-1\end{pmatrix}\right] = \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

The circular polarization states are: $R := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\i\end{pmatrix}$ $L := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-i\end{pmatrix}$ The appropriate operators are formed below:

$$RC := R\cdot\left(\overline{R}\right)^T \rightarrow \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\cdot i\\ \frac{1}{2}\cdot i & \frac{1}{2}\end{pmatrix} \qquad LC := L\cdot\left(\overline{L}\right)^T \rightarrow \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\cdot i\\ -\frac{1}{2}\cdot i & \frac{1}{2}\end{pmatrix} \qquad RLC := R\cdot\left(\overline{R}\right)^T - L\cdot\left(\overline{L}\right)^T \rightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

RLC is the angular momentum operator for photons. Below it is shown that |R> and |L> are eigenstates with eigenvalues of +1 and -1, respectively.

$$\text{eigenvals}(RLC) = \begin{pmatrix}1\\-1\end{pmatrix} \qquad RLC\cdot R = \begin{pmatrix}0.707\\0.707i\end{pmatrix} \qquad RLC\cdot L = \begin{pmatrix}-0.707\\0.707i\end{pmatrix} \qquad \frac{1}{\sqrt{2}} = 0.707$$

Now we can consider some of the measurements that Feynman discusses in his analysis of positronium annihilation. Because the photons in state Ψ are entangled the measurements of observers A and B are correlated. For example, if observers A and B both measure the circular polarization of their photons and compare their results they always agree that they have measured the **same** polarization state. Their composite expectation value is 1.

$$\Psi := \frac{i}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \qquad \left(\overline{\Psi}\right)^T\cdot\text{kronecker}(RLC, RLC)\cdot\Psi = 1$$

The identity operation, do nothing, is now needed: $I := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

But their individual results are a random sequence of +1 and -1 outcomes averaging to an expectation value of zero.

$$\left(\overline{\Psi}\right)^T\cdot\text{kronecker}(RLC, I)\cdot\Psi = 0 \qquad \left(\overline{\Psi}\right)^T\cdot\text{kronecker}(I, RLC)\cdot\Psi = 0$$

The probability that both observers will measure |R> or both measure |L> is 0.5. The probability that one will measure |R> and the other |L>, or vice versa is zero.

$$\left(\overline{\Psi}\right)^{T}\cdot\text{kronecker}(RC,RC)\cdot\Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker}(LC,LC)\cdot\Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker}(LC,RC)\cdot\Psi = 0$$

Now suppose the observers measure photon polarization in the vertical/horizontal basis. The appropriate eigenstates and measurement operators needed are shown below.

The eigenstates for plane polarization are: $V := \begin{pmatrix}1\\0\end{pmatrix} \qquad H := \begin{pmatrix}0\\1\end{pmatrix}$

The needed measurement operators are: $Vop := \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \qquad Hop := \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \qquad VH := \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

As **VH** is diagonal it is obvious that its eigenvalues are +1 and -1, and that V is the eigenstate with eigenvalue +1 and H is the eigenstate with eigenvalue -1.

$$VH\cdot V = \begin{pmatrix}1\\0\end{pmatrix} \qquad VH\cdot H = \begin{pmatrix}0\\-1\end{pmatrix}$$

Just as for the circular polarization measurements, the observers individual polarization measurements are totally random, but when they compare their results they find perfect anti-correlation, always observing the **opposite** polarization state.

$$\left(\overline{\Psi}\right)^{T}\cdot\text{kronecker}(VH,I)\cdot\Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker}(I,VH)\cdot\Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker}(VH,VH)\cdot\Psi = -1$$

Local classical reasoning is in disagreement with the highlighted result. Earlier it was demonstrated that the photons are either |L> or |R> polarized. However, suppose photon A is measured in the V-H basis and found to be |V>, and given that B is either |L> or |R>, which are superpositions of |V> and |H> (as shown below), measurement of B in the V-H basis should yield |V> 50% of the time and |H> 50% of the time. There should be no correlation between the A and B measurements in the V-H basis. The expectation value should be zero.

|R> $\quad \frac{1}{\sqrt{2}}\cdot(V + i\cdot H) = \begin{pmatrix}0.707\\0.707i\end{pmatrix}$ $\qquad$ |L> $\quad \frac{1}{\sqrt{2}}\cdot(V - i\cdot H) = \begin{pmatrix}0.707\\-0.707i\end{pmatrix}$

Arguing temporarily against non-local effects Feynman states: Surely you (A) cannot alter the physical state of *his* (B) photons by changing the kind of observation you make on *your* photons. No matter what measurements you make on yours, his must still be either RHC (|R>) or LHC (|L>).

However, because |R> and |L> are superpositions of |V> and |H>, the initial R-L wave function is also an entangled superposition in the V-H polarization basis.

$$|\Psi\rangle = \frac{i}{\sqrt{2}}\left[|V\rangle_A|H\rangle_B + |H\rangle_A|V\rangle_B\right] = \frac{i}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_A \otimes \begin{pmatrix}0\\1\end{pmatrix}_B + \begin{pmatrix}0\\1\end{pmatrix}_A \otimes \begin{pmatrix}1\\0\end{pmatrix}_B\right] = \frac{i}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

This wave function says a measurement of |V> at **A** collapses the wave function to |H> at **B** and a measurement of |H> at **A** collapses the wave function to |V> at **B**, in agreement with the highlighted expectation value. "A non-local interaction hooks up one location (A) with another (B) without crossing space, without decay, and without delay. A non-local event is, in short, unmediated, unmitigated and immediate." *Quantum Reality* by Nick Herbert, page 214.

$$\psi = \frac{1}{\sqrt{2}} \cdot \left(R_A \cdot R_B - L_A \cdot L_B\right) \left| \begin{array}{l} \text{substitute}, R_A = \frac{1}{\sqrt{2}} \cdot \left(V_A + i \cdot H_A\right) \\ \text{substitute}, R_B = \frac{1}{\sqrt{2}} \cdot \left(V_B + i \cdot H_B\right) \\ \text{substitute}, L_A = \frac{1}{\sqrt{2}} \cdot \left(V_A - i \cdot H_A\right) \\ \text{substitute}, L_B = \frac{1}{\sqrt{2}} \cdot \left(V_B - i \cdot H_B\right) \end{array} \right. \rightarrow \psi = \sqrt{2} \cdot \left(\frac{H_B \cdot V_A \cdot i}{2} + \frac{H_A \cdot V_B \cdot i}{2}\right)$$

The interference between the probability amplitudes after the RL to HV substitutions leading to the final state is shown below by a "hand" calculation. The unseen middle step.

$$\frac{1}{2\sqrt{2}}\left[V_A V_B + iV_A H_B + iH_A V_B - H_A H_B - V_A V_B + iV_A H_B + iH_A V_B + H_A H_B\right] = \frac{i}{\sqrt{2}}\left[V_A H_B + H_A V_B\right]$$

A summary of all possible measurement results:

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, RLC) \cdot \Psi = 1$

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, I) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, RC) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, LC) \cdot \Psi = -0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, VH) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, Vop) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, Hop) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(VH, VH) \cdot \Psi = -1$

$(\overline{\Psi})^T \cdot \text{kronecker}(VH, I) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(VH, Vop) \cdot \Psi = -0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(VH, Hop) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(VH, RC) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(VH, LC) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(RC, RC) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(LC, LC) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(RC, LC) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(RC, I) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(LC, I) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(Vop, Vop) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(Hop, Hop) \cdot \Psi = 0$

$(\overline{\Psi})^T \cdot \text{kronecker}(Vop, Hop) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(Vop, I) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(Hop, I) \cdot \Psi = 0.5$

$(\overline{\Psi})^T \cdot \text{kronecker}(RC, Vop) \cdot \Psi = 0.25$

$(\overline{\Psi})^T \cdot \text{kronecker}(RC, Hop) \cdot \Psi = 0.25$

$(\overline{\Psi})^T \cdot \text{kronecker}(LC, Vop) \cdot \Psi = 0.25$

$(\overline{\Psi})^T \cdot \text{kronecker}(LC, Hop) \cdot \Psi = 0.25$