Positronium Annihilation

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This tutorial provides an alternative analysis of the annihilation of positronium as presented in section 18-3 in Volume III of *The Feynman Lectures on Physics*. The calculations begin after the highlighted region which provides a justification for selecting an entangled singlet state for the two photons created in the decay event.

Positronium is an analog of the hydrogen atom in which the proton is replaced by a positron, the electron's anti-particle. The electron-positron pair undergoes annihilation in 10^{-10} seconds producing two γ-ray photons. Positronium's effective mass is 1/2, yielding a ground state energy (excluding the magnetic interactions between the spin 1/2 anti-particles) $E = -0.5\mu E_h = -0.25\ E_h$. Considering spin the ground state is four-fold degenerate, but this degeneracy is split by the magnetic spin-spin hyperfine interaction shown below. See "The Hyperfine Splitting in the Hydrogen Atom" for further detail.

$$\widehat{H}_{SpinSpin} = A\sigma^e \cdot \sigma^p = A\left(\sigma_x^e\sigma_x^p + \sigma_y^e\sigma_y^p + \sigma_z^e\sigma_z^p\right)$$

The spin-spin Hamiltonian has the following eigenvalues (top row) and eigenvectors (columns beneath the eigenvalues), showing a singlet ground state and triplet excited state. The electron-positron spin states are to the right of the table with their *m* quantum numbers, showing that the singlet (j = 0, m = 0) is a superposition state as is one of the triplet states (j = 1, m = 0). The parameter A is much larger for positronium than for the hydrogen atom because the positron has a much larger magnetic moment than the proton.

$$\begin{pmatrix} -3A & A & A & A \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} \left|++\right\rangle & m = 1 \\ \left|+-\right\rangle & m = 0 \\ \left|-+\right\rangle & m = 0 \\ \left|--\right\rangle & m = -1 \end{matrix}$$

Feynman shows that when the singlet ground state (J = 0, m = 0) annihilates, conservation of momentum requires that the photons emitted in opposite directions (A and B) must have the same circular polarization state, either both in the right or both in left circular state in their direction of motion. This leads to the following entangled superposition. The negative sign is required by parity conservation. The positronium ground state has negative parity (see above), therefore the final photon state must have negative parity.

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|R\right\rangle_A\left|R\right\rangle_B - \left|L\right\rangle_A\left|L\right\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\end{pmatrix}_A \otimes \begin{pmatrix}1\\i\end{pmatrix}_B - \begin{pmatrix}1\\-i\end{pmatrix}_A \otimes \begin{pmatrix}1\\-i\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\\i\\-1\end{pmatrix} - \begin{pmatrix}1\\-i\\-i\\-1\end{pmatrix}\right] = \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

The circular polarization states are: $R := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\i\end{pmatrix}$ $\quad L := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-i\end{pmatrix}$ The appropriate operators are formed below:

$$RC := R\cdot\left(\overline{R}\right)^T \rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\cdot i \\ \frac{1}{2}\cdot i & \frac{1}{2} \end{pmatrix} \qquad LC := L\cdot\left(\overline{L}\right)^T \rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\cdot i \\ -\frac{1}{2}\cdot i & \frac{1}{2} \end{pmatrix} \qquad RLC := R\cdot\left(\overline{R}\right)^T - L\cdot\left(\overline{L}\right)^T \rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

RLC is the angular momentum operator for photons. Below it is shown that |R> and |L> are eigenstates with eigenvalues of +1 and -1, respectively.

$$\text{eigenvals(RLC)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{RLC}\cdot\text{R} \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}\cdot i}{2} \end{pmatrix} \qquad \text{RLC}\cdot\text{L} \rightarrow \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}\cdot i}{2} \end{pmatrix}$$

Now we can consider some of the measurements that Feynman discusses in his analysis of positronium annihilation. Because the photons in state Ψ are entangled the measurements of observers A and B are correlated. For example, if observers A and B both measure the circular polarization of their photons and compare their results they always agree that they have measured the **same** polarization state. Their composite expectation value is 1.

$$\Psi := \frac{i}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \left(\overline{\Psi}\right)^{T} \cdot \text{kronecker(RLC, RLC)} \cdot \Psi = 1$$

The identity operation, do nothing, is now needed:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But their individual results are a random sequence of +1 and -1 outcomes averaging to an expectation value of zero.

$$\left(\overline{\Psi}\right)^{T} \cdot \text{kronecker(RLC, I)} \cdot \Psi = 0 \qquad \left(\overline{\Psi}\right)^{T} \cdot \text{kronecker(I, RLC)} \cdot \Psi = 0$$

The probability that both observers will measure |R> or both measure |L> is 0.5. The probability that one will measure |R> and the other |L>, or vice versa is zero.

$$\left(\overline{\Psi}\right)^{T} \cdot \text{kronecker(RC, RC)} \cdot \Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T} \cdot \text{kronecker(LC, LC)} \cdot \Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T} \cdot \text{kronecker(LC, RC)} \cdot \Psi = 0$$

Because |R> and |L> are superpositions of |V> and |H>, the photon wave function can also be written in the V-H plane polarization basis as is shown below. See the Appendix for an alternative justification.

$$\left|\Psi\right\rangle = \frac{i}{\sqrt{2}}\left[\left|V\right\rangle_A \left|H\right\rangle_B + \left|H\right\rangle_A \left|V\right\rangle_B\right] = \frac{i}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B\right] = \frac{i}{\sqrt{2}}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right] = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The eigenstates for plane polarization are:

$$V := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad H := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The appropriate measurement operators are:

$$\text{Vop} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{Hop} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{VH} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As **VH** is diagonal it is obvious that its eigenvalues are +1 and -1, and that V is the eigenstate with eigenvalue +1 and H is the eigenstate with eigenvalue -1.

$$\text{VH}\cdot\text{V} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{VH}\cdot\text{H} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Just as for the circular polarization measurements, the observers individual plane polarization measurements are totally random, but when they compare their results they find perfect anti-correlation, always observing the **opposite** polarization state.

$$\left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(VH, I)\cdot \Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(I, VH)\cdot \Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(VH, VH)\cdot \Psi = -1$$

$$\left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(Vop, Hop)\cdot \Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(Hop, Vop)\cdot \Psi = 0.5$$

$$\left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(Vop, Vop)\cdot \Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(Hop, Hop)\cdot \Psi = 0$$

If one observer measures circular polarization and the other measures plane polarization the expectation value is 0. In other words there is no correlation between the measurements.

$$\left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(RLC, VH)\cdot \Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot \text{kronecker}(VH, RLC)\cdot \Psi = 0$$

A realist believes that all observables have definite values independent of measurement and that measurement of one variable doesn't affect the value of another variable. Such a person might construct the following table which assigns specific polarization states to the photons in the R-L and V-H bases to explain the quantum mechanical calculations performed above.

PhotonA	'	PhotonB	'	RL(A)·RL(B)	VH(A)·VH(B)	RL(A)·VH(B)	VH(A)·RL(B)
R·V	'	R·H	'	1	−1	−1	1
L·V	'	L·H	'	1	−1	1	−1
R·H	'	R·V	'	1	−1	1	−1
L·H	'	L·V	'	1	−1	−1	1
Expectation	'	Value	'	1	−1	0	0

However, the quantum theorist objects that the operators representing rectilinear and circular polarization do not commute, which means that they represent incompatible observables which cannot simultaneously occupy well-defined states.

$$RLC\cdot VH - VH\cdot RLC = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$$

As shown below |R> and |L> are superpositions of |V> and |H> and vice versa. This is another way of demonstrating that a photon cannot be in a well-defined circular polarization state, say |R>, and at the same time be definitely either |V> or |H>. |R> is a superposition of |V> and |H> and therefore its plane polarization state is completely undetermined. Thus the photon states in the table proposed by the realist are not valid from the quantum mechanical perspective. They have, therefore, no explanatory validity.

|R> |L> |V> |H>

$$\frac{1}{\sqrt{2}}\cdot(V + i\cdot H) \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}\cdot i}{2} \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\cdot(V - i\cdot H) \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}\cdot i}{2} \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\cdot(L + R) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{i}{\sqrt{2}}\cdot(L - R) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Appendix

Transforming Ψ from the R-L basis to the V-H basis using the superpositions above.

$$\psi = \frac{1}{\sqrt{2}}\cdot\left(R_A\cdot R_B - L_A\cdot L_B\right) \left| \begin{array}{l} \text{substitute}, R_A = \frac{1}{\sqrt{2}}\cdot\left(V_A + i\cdot H_A\right) \\ \text{substitute}, R_B = \frac{1}{\sqrt{2}}\cdot\left(V_B + i\cdot H_B\right) \\ \text{substitute}, L_A = \frac{1}{\sqrt{2}}\cdot\left(V_A - i\cdot H_A\right) \\ \text{substitute}, L_B = \frac{1}{\sqrt{2}}\cdot\left(V_B - i\cdot H_B\right) \\ \text{simplify} \end{array} \right. \rightarrow \psi = \sqrt{2}\cdot\left(\frac{H_A\cdot V_B}{2} + \frac{H_B\cdot V_A}{2}\right)\cdot i$$

I thought it would be interesting to look at calculations that included measurement in the diagonal-slant rectilinear basis.

The diagonal and slant eigenvectors: $D := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}$ $\quad S := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-1\end{pmatrix}$ $\quad \frac{D-S}{\sqrt{2}} = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad \frac{D+S}{\sqrt{2}} = \begin{pmatrix}1\\0\end{pmatrix}$

The original state function in the diagonal-slant basis:

$$\psi = \frac{i}{\sqrt{2}}\cdot\left(H_A\cdot V_B + V_A\cdot H_B\right) \left| \begin{array}{l} \text{substitute}, H_A = \frac{1}{\sqrt{2}}\cdot\left(D_A - S_A\right) \\ \text{substitute}, H_B = \frac{1}{\sqrt{2}}\cdot\left(D_B - S_B\right) \\ \text{substitute}, V_A = \frac{1}{\sqrt{2}}\cdot\left(D_A + S_A\right) \\ \text{substitute}, V_B = \frac{1}{\sqrt{2}}\cdot\left(D_B + S_B\right) \\ \text{simplify} \end{array} \right. \rightarrow \psi = \sqrt{2}\cdot\left(\frac{D_A\cdot D_B}{2} - \frac{S_A\cdot S_B}{2}\right)\cdot i$$

$$DS := \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad Dd := \frac{1}{2}\cdot\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \qquad Ds := \frac{1}{2}\cdot\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} \qquad D\cdot D^T - S\cdot S^T = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$\left(\overline{\Psi}\right)^T\cdot\text{kronecker}(DS, DS)\cdot\Psi = 1 \qquad \left(\overline{\Psi}\right)^T\cdot\text{kronecker}(Dd, Dd)\cdot\Psi = 0.5 \qquad \left(\overline{\Psi}\right)^T\cdot\text{kronecker}(Ds, Ds)\cdot\Psi = 0.5$$

$$\left(\overline{\Psi}\right)^T\cdot\text{kronecker}(Dd, Ds)\cdot\Psi = 0 \qquad \left(\overline{\Psi}\right)^T\cdot\text{kronecker}(DS, RLC)\cdot\Psi = 0 \qquad \left(\overline{\Psi}\right)^T\cdot\text{kronecker}(DS, VH)\cdot\Psi = 0$$

$$VH\cdot DS - DS\cdot VH = \begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix} \qquad RLC\cdot DS - DS\cdot RLC = \begin{pmatrix}-2i & 0\\0 & 2i\end{pmatrix}$$

All of the states below give expectation values that agree with quantum mechanics, but are not permissible in quantum mechanics because the operators do not commute.

PhotonA	PhotonB	RL(A)·RL(B)	VH(A)·VH(B)	RL(A)·VH(B)	DS(A)·DS(B)	RL(A)·DS(B)	VH(A)·DS(B)
RVD	RHD	1	–1	–1	1	1	1
RVS	RHS	1	–1	–1	1	–1	–1
LVD	LHD	1	–1	1	1	–1	1
LVS	LHS	1	–1	1	1	1	–1
RHD	RVD	1	–1	1	1	1	–1
RHS	RVS	1	–1	1	1	–1	1
LHD	LVD	1	–1	–1	1	–1	–1
LHS	LVS	1	–1	–1	1	1	1
Expectation	Value	1	–1	0	1	0	0

$$E(\theta) := \left(\overline{\Psi}\right)^T \cdot \begin{pmatrix} \cos(2\cdot\theta) & \sin(2\cdot\theta) & 0 & 0 \\ \sin(2\cdot\theta) & -\cos(2\cdot\theta) & 0 & 0 \\ 0 & 0 & -\cos(2\cdot\theta) & -\sin(2\cdot\theta) \\ 0 & 0 & -\sin(2\cdot\theta) & \cos(2\cdot\theta) \end{pmatrix} \cdot \Psi \rightarrow -\cos(2\cdot\theta)$$