Positronium Annihilation

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The following provides an alternative mathematical analysis of the annihilation of positronium as presented in section 18-3 in Volume III of *The Feynman Lectures on Physics*.

Positronium is an analog of the hydrogen atom in which the proton is replaced by a positron, the electron's anti-particle. The electron-positron pair undergoes annihilation in 10^{-10} seconds producing two γ-ray photons. Positronium's effective mass is 1/2, yielding a ground state energy (excluding the magnetic interactions between the spin 1/2 anti-particles) $E = -0.5\mu E_h = -0.25\ E_h$. Considering spin the ground state is four-fold degenerate, but this degeneracy is split by the magnetic spin-spin hyperfine interaction shown below. See "The Hyperfine Splitting in the Hydrogen Atom" for further detail.

$$\widehat{H}_{SpinSpin} = A\sigma^e \cdot \sigma^p = A\left(\sigma_x^e\sigma_x^p + \sigma_y^e\sigma_y^p + \sigma_z^e\sigma_z^p\right)$$

The spin-spin Hamiltonian has the following eigenvalues (top row) and eigenvectors (columns beneath the eigenvalues), showing a singlet ground state and triplet excited state. The electron-positron spin states are to the right of the table with their *m* quantum numbers, showing that the singlet (j = 0, m = 0) is a superposition state as is one of the triplet states (j = 1, m = 0). The parameter A is much larger for positronium than for the hydrogen atom because the positron has a much larger magnetic moment than the proton.

$$\begin{pmatrix} -3A & A & A & A \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} \\ |++\rangle & m = 1 \\ |+-\rangle & m = 0 \\ |-+\rangle & m = 0 \\ |--\rangle & m = -1 \end{matrix}$$

Feynman shows that when the singlet ground state (J = 0, m = 0) annihilates, conservation of momentum requires that the photons emitted in opposite directions (A and B) must have the same circular polarization state, either both in the right or both in left circular state in their direction of motion. This leads to the following entangled superposition. The negative sign is required by parity conservation. The positronium ground state has negative parity (see above), therefore the final photon state must have negative parity.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|R\rangle_A|R\rangle_B - |L\rangle_A|L\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\end{pmatrix}_A \otimes \begin{pmatrix}1\\i\end{pmatrix}_B - \begin{pmatrix}1\\-i\end{pmatrix}_A \otimes \begin{pmatrix}1\\-i\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\\i\\-1\end{pmatrix} - \begin{pmatrix}1\\-i\\-i\\-1\end{pmatrix}\right] = \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

The circular polarization states are: $R := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\i\end{pmatrix}$ $L := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-i\end{pmatrix}$ The appropriate operators are formed below:

$$RC := R\cdot\left(\overline{R}\right)^T \rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\cdot i \\ \frac{1}{2}\cdot i & \frac{1}{2} \end{pmatrix} \qquad LC := L\cdot\left(\overline{L}\right)^T \rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\cdot i \\ -\frac{1}{2}\cdot i & \frac{1}{2} \end{pmatrix} \qquad RLC := R\cdot\left(\overline{R}\right)^T - L\cdot\left(\overline{L}\right)^T \rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

RLC is the angular momentum operator for photons. Below it is shown that |R> and |L> are eigenstates with eigenvalues of +1 and -1, respectively.

$$\text{eigenvals(RLC)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{RLC}\cdot\text{R} \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}\cdot i}{2} \end{pmatrix} \qquad \text{RLC}\cdot\text{L} \rightarrow \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}\cdot i}{2} \end{pmatrix}$$

Now we can consider some of the measurements that Feynman discusses in his analysis of positronium annihilation. Because the photons in state Ψ are entangled the measurements of observers A and B are correlated. For example, if observers A and B both measure the circular polarization of their photons and compare their results they always agree that they have measured the **same** polarization state. Their composite expectation value is 1.

$$\Psi := \frac{i}{\sqrt{2}}\cdot\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker(RLC, RLC)}\cdot\Psi = 1$$

The identity operation, do nothing, is now needed:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But their individual results are a random sequence of +1 and -1 outcomes averaging to an expectation value of zero.

$$\left(\overline{\Psi}\right)^{T}\cdot\text{kronecker(RLC, I)}\cdot\Psi = 0 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker(I, RLC)}\cdot\Psi = 0$$

The probability that both observers will measure |R> or both measure |L> is 0.5. The probability that one will measure |R> and the other |L>, or vice versa is zero.

$$\left(\overline{\Psi}\right)^{T}\cdot\text{kronecker(RC, RC)}\cdot\Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker(LC, LC)}\cdot\Psi = 0.5 \qquad \left(\overline{\Psi}\right)^{T}\cdot\text{kronecker(LC, RC)}\cdot\Psi = 0$$

Because |R> and |L> are superpositions of |V> and |H>, the photon wave function can also be written in the V-H plane polarization basis as is shown below. See the Appendix for an alternative justification.

$$|\Psi\rangle = \frac{i}{\sqrt{2}}\left[|V\rangle_A |H\rangle_B + |H\rangle_A |V\rangle_B\right] = \frac{i}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B\right] = \frac{i}{\sqrt{2}}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right] = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The eigenstates for plane polarization are:

$$V := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad H := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The appropriate measurement operators are:

$$\text{Vop} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{Hop} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{VH} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

As **VH** is diagonal it is obvious that its eigenvalues are +1 and -1, and that V is the eigenstate with eigenvalue +1 and H is the eigenstate with eigenvalue -1.

$$\text{VH}\cdot\text{V} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{VH}\cdot\text{H} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Just as for the circular polarization measurements, the observers individual plane polarization measurements are totally random, but when they compare their results they find perfect anti-correlation, always observing the **opposite** polarization state.

$$(\overline{\Psi})^T \cdot \text{kronecker}(VH, I) \cdot \Psi = 0 \qquad (\overline{\Psi})^T \cdot \text{kronecker}(I, VH) \cdot \Psi = 0 \qquad (\overline{\Psi})^T \cdot \text{kronecker}(VH, VH) \cdot \Psi = -1$$

$$(\overline{\Psi})^T \cdot \text{kronecker}(Vop, Hop) \cdot \Psi = 0.5 \qquad (\overline{\Psi})^T \cdot \text{kronecker}(Hop, Vop) \cdot \Psi = 0.5$$

$$(\overline{\Psi})^T \cdot \text{kronecker}(Vop, Vop) \cdot \Psi = 0 \qquad (\overline{\Psi})^T \cdot \text{kronecker}(Hop, Hop) \cdot \Psi = 0$$

If one observer measures circular polarization and the other measures plane polarization the expectation value is 0. In other words there is no correlation between the measurements.

$$(\overline{\Psi})^T \cdot \text{kronecker}(RLC, VH) \cdot \Psi = 0 \qquad (\overline{\Psi})^T \cdot \text{kronecker}(VH, RLC) \cdot \Psi = 0$$

Classical reasoning (according to Feynman) is in disagreement with the highlighted result. Earlier it was demonstrated that the photons are either |L> or |R> polarized. However, suppose photon A is measured in the V-H basis and found to be |V>, and given that B is either |L> or |R>, which are superpositions of |V> and |H> (see Appendix), measurement of B in the V-H basis should yield |V> 50% of the time and |H> 50% of the time. There should be no correlation between the A and B measurements. The expectation value should be zero.

Feynman put it this way (parenthetical material added):

> Surely you (A) cannot alter the physical state of *his* (B) photons by changing the kind of observation you make on *your* photons. No matter what measurements you make on yours, his must still be either RHC (|R>) or LHC (|L>).

But according to quantum mechanics the photons are entangled in the R-L and V-H bases as shown above, and therefore measurement of |V> at **A** collapses the wave function to |H> at **B**.

The highlighted prediction is confirmed experimentally leading to the conclusion that reasoning classically in this manner about the photons created in positronium annihilation is not valid.

While this analysis of positronium annihilation clarifies the conflict between quantum theory and classical realism, it does not lead to an experimental adjudication of the disagreement. In 1964 John Bell demonstrated that entangled systems, like the positronium decay products, could be used to decide the conflict one way or the other empirically. As is well known the subsequent experimental work based on Bell's theorem decided the conflict between the two views in favor of quantum theory.

Appendix

The relationships between plane and circularly polarized light.

|R>	|L>	|V>	|H>
$\frac{1}{\sqrt{2}}\cdot(V + i\cdot H) \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}\cdot i}{2} \end{pmatrix}$	$\frac{1}{\sqrt{2}}\cdot(V - i\cdot H) \rightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}\cdot i}{2} \end{pmatrix}$	$\frac{1}{\sqrt{2}}\cdot(L + R) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\frac{i}{\sqrt{2}}\cdot(L - R) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Transforming Ψ from the R-L basis to the V-H basis using the superpositions above.

$$\psi = \frac{1}{\sqrt{2}}\cdot\left(R_A\cdot R_B - L_A\cdot L_B\right) \left| \begin{array}{l} \text{substitute}, R_A = \frac{1}{\sqrt{2}}\cdot\left(V_A + i\cdot H_A\right) \\ \text{substitute}, R_B = \frac{1}{\sqrt{2}}\cdot\left(V_B + i\cdot H_B\right) \\ \text{substitute}, L_A = \frac{1}{\sqrt{2}}\cdot\left(V_A - i\cdot H_A\right) \\ \text{substitute}, L_B = \frac{1}{\sqrt{2}}\cdot\left(V_B - i\cdot H_B\right) \\ \text{simplify} \end{array} \right. \rightarrow \psi = \sqrt{2}\cdot\left(\frac{H_A\cdot V_B}{2} + \frac{H_B\cdot V_A}{2}\right)\cdot i$$