

Examining the Local States of an Entangled Bipartite Superposition

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Art Hobson recently posted "Implications of bipartite interferometry for the measurement problem" at [arXiv:1301.1673](https://arxiv.org/abs/1301.1673). In this post he provides the following interpretation (modified slightly by F. Rioux) of entangled bipartite spin systems. The mathematics below is provided in support of Hobson's analysis.

When a bipartite system is in an entangled superposition, its subsystems are not in superpositions but are instead mixed states with each subsystem in a definite, but unknown state. An entangled spin is always in its **local** state, the state described by its **reduced density operator**, because this is the state actually detected by an observer of the spin. The entangled state is a **global** superposition of spin correlations, not a superposition of **local** spin states.

The **reduced density operator** of an entangled spin is obtained by tracing (averaging) the **total density operator** over the other spin. This procedure shows that a spin's **reduced density operator** is diagonal indicating a classical mixed state - it has been stripped of its off-diagonal interference terms.

An entangled bipartite superposition: $|\Psi\rangle = \frac{1}{\sqrt{2}} [|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle]$ where $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Its total density operator: $\widehat{\rho}_{12} = |\Psi\rangle\langle\Psi| = \frac{1}{2} [|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle][\langle\downarrow_2|\langle\uparrow_1| - \langle\uparrow_2|\langle\downarrow_1|] = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The total density operator written in two equivalent forms:

$$\widehat{\rho}_{12} = |\Psi\rangle\langle\Psi| = \frac{1}{2} [|\uparrow_1\rangle \otimes |\downarrow_2\rangle \langle\downarrow_2| \otimes \langle\uparrow_1| - |\uparrow_1\rangle \otimes |\downarrow_2\rangle \langle\uparrow_2| \otimes \langle\downarrow_1| - |\downarrow_1\rangle \otimes |\uparrow_2\rangle \langle\downarrow_2| \otimes \langle\uparrow_1| + |\downarrow_1\rangle \otimes |\uparrow_2\rangle \langle\uparrow_2| \otimes \langle\downarrow_1|]$$

$$\rho_{12} := \frac{1}{2} \cdot \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot (0 \ 1 \ 0 \ 0) - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot (0 \ 0 \ 1 \ 0) - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot (0 \ 1 \ 0 \ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot (0 \ 0 \ 1 \ 0) \right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\widehat{\rho}_{12} = |\Psi\rangle\langle\Psi| = \frac{1}{2} [|\uparrow_1\rangle\langle\uparrow_1| \otimes |\downarrow_2\rangle\langle\downarrow_2| - |\uparrow_1\rangle\langle\downarrow_1| \otimes |\downarrow_2\rangle\langle\uparrow_2| - |\downarrow_1\rangle\langle\uparrow_1| \otimes |\uparrow_2\rangle\langle\downarrow_2| + |\downarrow_1\rangle\langle\downarrow_1| \otimes |\uparrow_2\rangle\langle\uparrow_2|]$$

$$\rho_{12} := \frac{1}{2} \cdot \left[\text{kroncker} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1 \ 0), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (0 \ 1) \right] - \text{kroncker} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (0 \ 1), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (1 \ 0) \right] \dots \right] + \left[-\text{kroncker} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (1 \ 0), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (0 \ 1) \right] + \text{kroncker} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (0 \ 1), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1 \ 0) \right] \right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Calculation of the reduced density operator of spin 1 by tracing the total density operator over spin 2:

$$\widehat{\rho}_1 = \frac{1}{2} [\langle\uparrow_2|\Psi\rangle\langle\Psi|\uparrow_2\rangle + \langle\downarrow_2|\Psi\rangle\langle\Psi|\downarrow_2\rangle] = \frac{1}{2} [|\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_1\rangle\langle\uparrow_1|] = \frac{1}{2} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) \right] = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_1 := \frac{1}{2} \cdot \left[\begin{array}{l} \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (1 \ 0) \cdot \text{tr} \left[\left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (0 \ 1) \right] - \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (0 \ 1) \cdot \text{tr} \left[\left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (1 \ 0) \right] \dots \\ + \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (1 \ 0) \cdot \text{tr} \left[\left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (0 \ 1) \right] + \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (0 \ 1) \cdot \text{tr} \left[\left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (1 \ 0) \right] \end{array} \right] \rightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Calculation of the reduced density operator of spin 2 by tracing the total density operator over spin 1:

$$\widehat{\rho}_2 = \frac{1}{2} [\langle \uparrow_1 | \Psi \rangle \langle \Psi | \uparrow_1 \rangle + \langle \downarrow_1 | \Psi \rangle \langle \Psi | \downarrow_1 \rangle] = \frac{1}{2} [\langle \downarrow_2 | \langle \downarrow_2 | + | \uparrow_2 \rangle \langle \uparrow_2 |] = \frac{1}{2} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) \right] = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_2 := \frac{1}{2} \cdot \left[\begin{array}{l} \text{tr} \left[\left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (1 \ 0) \right] \cdot \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (0 \ 1) - \text{tr} \left[\left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (0 \ 1) \right] \cdot \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (1 \ 0) \dots \\ + \text{tr} \left[\left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (1 \ 0) \right] \cdot \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (0 \ 1) + \text{tr} \left[\left(\begin{array}{c} 0 \\ 1 \end{array} \right) \cdot (0 \ 1) \right] \cdot \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \cdot (1 \ 0) \end{array} \right] \rightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Now let's flesh this out with specific calculations. Below we have the entangled singlet spin state, its total density operator, and the reduced density operators for the individual spins.

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \rho_{12} := \Psi \cdot \Psi^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho_1 := \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_2 := \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Next we need the operator required for spin measurements in the z-direction and the identity operator, do nothing.

$$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we calculate expectation values for some spin measurements in the z-direction. As shown below this can be accomplished in two ways, the latter on the right, uses the trace function.

$$\langle \Psi | \hat{O} | \Psi \rangle = \sum_i \langle \Psi | \hat{O} | i \rangle \langle i | \Psi \rangle = \sum_i \langle i | \Psi \rangle \langle \Psi | \hat{O} | i \rangle = \text{Trace}(|\Psi\rangle\langle\Psi| \hat{O}) \quad \text{where} \quad \sum_i |i\rangle\langle i| = \text{Identity}$$

If we measure the z-component of both spins we find perfect anti-correlation which is expected for an entangled singlet state.

$$\Psi^T \cdot \text{kroncker}(\sigma_z, \sigma_z) \Psi = -1 \quad \text{tr}(\Psi \cdot \Psi^T \text{kroncker}(\sigma_z, \sigma_z)) = -1$$

However, if only one spin is measured the expectation value is 0.

$$\Psi^T \cdot \text{kroncker}(\sigma_z, I) \Psi = 0 \quad \text{tr}(\Psi \cdot \Psi^T \text{kroncker}(\sigma_z, I)) = 0 \quad \Psi^T \cdot \text{kroncker}(I, \sigma_z) \Psi = 0 \quad \text{tr}(\Psi \cdot \Psi^T \text{kroncker}(I, \sigma_z)) = 0$$

These results are consistent with the following calculations, which use the reduced density operators for the individual spins. In other words it assumes the spins are in local mixed states, as was asserted previously. The trace function is required for the calculation of expectation values of mixed states.

$$\text{tr}(\rho_1 \cdot \sigma_z) = 0 \quad \text{tr}(\rho_2 \cdot \sigma_z) = 0$$

To repeat, the entangled state Ψ is a **global** superposition of spin correlations, not a superposition of **local** spin states.