Another Example of Teleportation Using Quantum Gates

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In this example of teleportation using quantum gates we will dispense with Alice, Bob and Carol, and talk instead about transferring the cubit |Ψ > in the quantum circuit below from the first wire to the third wire.

Initial	1	2	3	4		Final
$\|\Psi\rangle$	$\cdots$	$\cdots$	$\bullet$	$\boxed{H}$	$\triangleright$	Measure $\|a\rangle$ 0 or 1
			$\|$			Bell state measurement
$\|0\rangle$	$\boxed{H}$	$\bullet$	$\oplus$	$\cdots$	$\triangleright$	Measure $\|b\rangle$ 0 or 1
		$\|$				
$\|0\rangle$	$\cdots$	$\oplus$	$\cdots$	$\cdots$	$\triangleright$	$X^b Z^a \rightarrow \|\Psi\rangle$

The quantum teleportation circuit is adapted from the one shown on page 226 of *The Quest for the Quantum Computer* by Julian Brown. This tutorial also draws on Brad Rubin's "Quantum Teleportation" at the Wolfram Demonstration Project: http:demonstrations.wolfram.com/QuantumTeleportation/.

In the matrix version of quantum mechanics, vectors represent states and matrices represent operators or, in this application, quantum gates. Quantum gates are required to be unitary matrices.

The necessary quantum bits or qubit states are:

Base states: A superposition of base states:

$$0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{where} \quad (|\alpha|)^2 + (|\beta|)^2 = 1$$

The identity operator and the following quantum gates are required to calculate the result of the teleportation circuit displayed above up to the point of the measurements on qubits 1 and 2.

Identity Hadamard gate Controlled-NOT gate

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We begin by using $\begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} = \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix}$ as the input qubit on wire 1. This choice means that the three-qubit initial state is,

$$\begin{pmatrix}\sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{2}{3}}\\0\\0\\0\\ \sqrt{\frac{1}{3}}\\0\\0\\0\end{pmatrix} \qquad \Psi := \begin{pmatrix}\sqrt{\frac{2}{3}}\\0\\0\\0\\ \sqrt{\frac{1}{3}}\\0\\0\\0\end{pmatrix}$$

The following operations on the initial state yield the state (Ψ') prior to the measurements on qubits 1 and 2. In the Mathcad programming environment, *kronecker* is the command for matrix tensor multiplication.

$$Step1 := kronecker(I, kronecker(H, I)) \qquad Step2 := kronecker(I, CNOT) \qquad Step3 := kronecker(CNOT, I)$$

$$Step4 := kronecker(H, kronecker(I, I))$$

$$\Psi' := Step4 \cdot Step3 \cdot Step2 \cdot Step1 \cdot \Psi \qquad \Psi'^{T} = (0.408\ \ 0.289\ \ 0.289\ \ 0.408\ \ 0.408\ \ -0.289\ \ -0.289\ \ 0.408)$$

There are four possible measurement outcomes on qubits 1 and 2 in the z-basis: |00>, |01>, |10 > and |11 >. The projection operators for |0 > and |1 > are given below.

Projection operator for |0 >: $\begin{pmatrix}1\\0\end{pmatrix}\cdot(1\ \ 0)=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ Projection operator for |1 >: $\begin{pmatrix}0\\1\end{pmatrix}\cdot(0\ \ 1)=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

As shown below, depending on the measurement results the following unitary operations are required to complete the transfer of $\begin{pmatrix}0.816\\0.577\end{pmatrix}$ to the third wire: I, σ_x, σ_z, and $\sigma_z\sigma_x$. The needed Pauli operators are:

$$\sigma_x := \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \sigma_z := \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Measurement result for qubits 1 and 2	Final 3-qubit state	Required operation
$2\cdot kronecker\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}, kronecker\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}, I\right]\right]\cdot\Psi' = \begin{pmatrix}0.816\\0.577\\0\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0.816\\0.577\end{pmatrix}$	$I\cdot\begin{pmatrix}0.816\\0.577\end{pmatrix}=\begin{pmatrix}0.816\\0.577\end{pmatrix}$

$$2\cdot\mathrm{kronecker}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\mathrm{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},I\right]\right]\cdot\Psi' = \begin{pmatrix}0\\0\\0.577\\0.816\\0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0.577\\0.816\end{pmatrix} \qquad \sigma_x\cdot\begin{pmatrix}0.577\\0.816\end{pmatrix} = \begin{pmatrix}0.816\\0.577\end{pmatrix}$$

$$2\cdot\mathrm{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\mathrm{kronecker}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},I\right]\right]\cdot\Psi' = \begin{pmatrix}0\\0\\0\\0\\0.816\\-0.577\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0.816\\-0.577\end{pmatrix} \qquad \sigma_z\cdot\begin{pmatrix}0.816\\-0.577\end{pmatrix} = \begin{pmatrix}0.816\\0.577\end{pmatrix}$$

$$2\cdot\mathrm{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\mathrm{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},I\right]\right]\cdot\Psi' = \begin{pmatrix}0\\0\\0\\0\\0\\0\\-0.577\\0.816\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}-0.577\\0.816\end{pmatrix} \qquad \sigma_z\cdot\sigma_x\cdot\begin{pmatrix}-0.577\\0.816\end{pmatrix} = \begin{pmatrix}0.816\\0.577\end{pmatrix}$$

In the interest of relating this example to other examples of quantum teleportation, it is pointed out that the first two steps in the diagram above create an entangled Bell state involving qubits 2 and 3.

$$\mathrm{CNOT}\cdot\mathrm{kronecker}(H,I)\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0.707\\0\\0\\0.707\end{pmatrix}$$