Teleportation Using Quantum Gates

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Implementation of the following 8-step circuit using quantum gates teleports cubit $|\Psi\rangle$ from the top wire to the bottom wire. The circuit can be found on page 226 of Julian Brown's *The Quest for the Quantum Computer*. A final Bell state measurement on the top wires teleports $|\Psi\rangle$ to the bottom wire.

Initial	1	2	3	4	5	6	7	8		Final
$\lvert\Psi\rangle$	···	···	•	H	···	···	•	···	▷	Measure, 0 or 1
			|				|			Bell state measurement
$\lvert 0\rangle$	H	•	⊕	···	•	···	|	···	▷	Measure, 0 or 1
		|			|		|			
$\lvert 0\rangle$	···	⊕	···	···	⊕	H	⊕	H	▷	$\lvert\Psi\rangle$

In the matrix version of quantum mechanics, vectors represent states and matrices represent operators or, in this application, quantum gates. Quantum gates are required to be unitary matrices.

The necessary quantum bits or qubit states are:

Base states: A superposition of base states:

$$0 = \begin{pmatrix}1\\0\end{pmatrix} \qquad 1 = \begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\cdot\begin{pmatrix}1\\0\end{pmatrix} + \beta\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

where $(|\alpha|)^2 + (|\beta|)^2 = 1$

The identity operator and the following quantum gates are required to calculate the result of the teleportation circuit displayed above.

Identity

$$I := \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Hadamard gate

$$H := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

Controlled-NOT gate

$$CNOT := \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}$$

Step-7 Controlled-NOT gate

$$CnNOT := \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

Using $\begin{pmatrix}\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{3}}\end{pmatrix} = \begin{pmatrix}0.816\\0.577\end{pmatrix}$ as the input qubit, the three-qubit initial state is,

$$\begin{pmatrix}\sqrt{2/3}\\ \sqrt{1/3}\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\sqrt{2/3}\\0\\0\\0\\ \sqrt{1/3}\\0\\0\\0\end{pmatrix} \qquad \Psi := \begin{pmatrix}\sqrt{\frac{2}{3}}\\0\\0\\0\\ \sqrt{\frac{1}{3}}\\0\\0\\0\end{pmatrix}$$

The matrix operators required for the steps of the teleportation circuit are now constructed. In the Mathcad programming environment, *kronecker* is the command for matrix tensor multiplication.

$$\text{Step1} := \text{kronecker}(I, \text{kronecker}(H, I)) \qquad \text{Step2} := \text{kronecker}(I, \text{CNOT}) \qquad \text{Step3} := \text{kronecker}(\text{CNOT}, I)$$

$$\text{Step4} := \text{kronecker}(H, \text{kronecker}(I, I)) \qquad \text{Step5} := \text{kronecker}(I, \text{CNOT}) \qquad \text{Step6} := \text{kronecker}(I, \text{kronecker}(I, H))$$

$$\text{Step7} := \text{CnNOT} \qquad \text{Step8} := \text{kronecker}(I, \text{kronecker}(I, H))$$

$$\text{QuantumCircuit} := \text{Step8}\cdot\text{Step7}\cdot\text{Step6}\cdot\text{Step5}\cdot\text{Step4}\cdot\text{Step3}\cdot\text{Step2}\cdot\text{Step1}$$

$$\Psi' := \text{QuantumCircuit}\cdot\Psi \qquad \Psi'^{T} = \begin{pmatrix}0.408 & 0.289 & 0.408 & 0.289 & 0.408 & 0.289 & 0.408 & 0.289\end{pmatrix}$$

A Bell state measurement on the top wires in the computational basis collapses the wave function and achieves the desired teleportation no matter what the actual measurement results are. There are, of course, four possibilites |0>|0>, |0>|1>, |1>|0> and |1>|1>. All result in $|\Psi>$ on the bottom wire without further action as is now shown. (The calculations are multiplied by a factor 2 to normalize the result.)

Measurement operator for |0 >: $\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

Measurement operator for |1 >: $\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

$$\begin{bmatrix} a' & b' & c' \\ \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0.816\\0.577\end{pmatrix}\end{bmatrix} \qquad \begin{bmatrix} a' & b' & c' \\ \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0.816\\0.577\end{pmatrix}\end{bmatrix}$$

$$2\cdot\text{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \text{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, I\right]\right]\cdot\Psi' = \begin{pmatrix}0.816\\0.577\\0\\0\\0\\0\\0\\0\end{pmatrix} \qquad 2\cdot\text{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \text{kronecker}\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, I\right]\right]\cdot\Psi' = \begin{pmatrix}0\\0\\0.816\\0.577\\0\\0\\0\\0\end{pmatrix}$$

$$\begin{bmatrix} a' & b' & c' \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix} \end{bmatrix}$$

$$2 \cdot \text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{kronecker}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, I\right]\right] \cdot \Psi' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.816 \\ 0.577 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} a' & b' & c' \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix} \end{bmatrix}$$

$$2 \cdot \text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, I\right]\right] \cdot \Psi' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.816 \\ 0.577 \end{pmatrix}$$