Teleportation as a Quantum Computation

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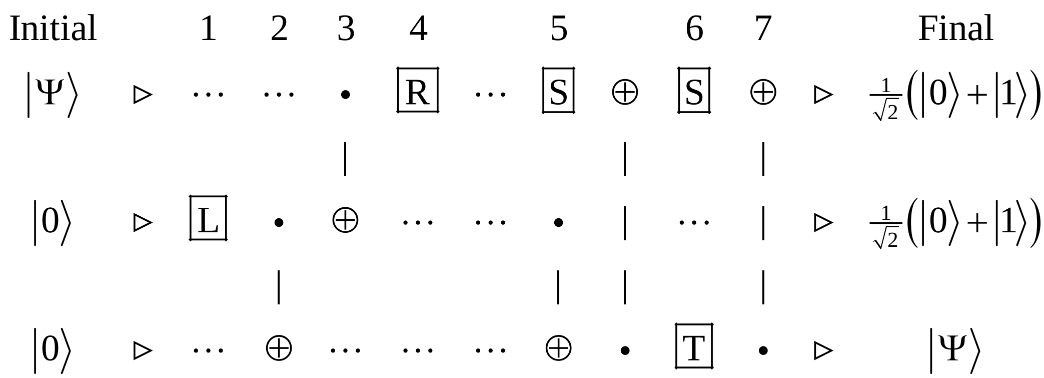
This tutorial works through the following teleportation circuit provided by Gilles Brassard in "Teleportation as a Quantum Computation" (arXiv:quant-ph/9605035v1). The computational methodology employed here is similar to that used in the other teleportation examples given in this series of tutorials.



The necessary quantum bits or qubit states are:

Base states: Superposition of base states:

$$0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{where} \quad (|\alpha|)^2 + (|\beta|)^2 = 1$$

Using $\begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} = \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix}$ as the teleportee (|Ψ>), Brassard's circuit yields the final states shown in the circuit above. In other words |Ψ > is teleported from the first wire on the left to the third wire on the right.

Initial state: $\begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} \\ 0 \\ 0 \\ 0 \\ \sqrt{1/3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Final state: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix} = \begin{pmatrix} \sqrt{1/6} \\ \sqrt{1/12} \\ \sqrt{1/6} \\ \sqrt{1/12} \\ \sqrt{1/6} \\ \sqrt{1/12} \\ \sqrt{1/6} \\ \sqrt{1/12} \end{pmatrix}$

The identity operator and the following quantum gates are required to calculate the result of the teleportation circuit. **L** and **R** are single qubit rotations, **S** and **T** are single qubit phase shifts. The other gates (CNOT, CnNOT, and ICnNOT) are well-known in quantum circuitry.

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad L := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad R := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad S := \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \qquad T := \begin{pmatrix} -1 & 0 \\ 0 & -i \end{pmatrix}$$

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad CnNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad ICnNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$Step1 := kronecker(I, kronecker(L, I))$ $\quad$ $Step2 := kronecker(I, CNOT)$ $\quad$ $Step3 := kronecker(CNOT, I)$

$Step4 := kronecker(R, kronecker(I, I))$ $\quad$ $Step5 := kronecker(S, CNOT)$ $\quad$ $Step6 := ICnNOT$

$Step7 := kronecker(S, kronecker(I, T))$ $\quad$ $Step8 := ICnNOT$

$$\Psi_{final} := Step8 \cdot Step7 \cdot Step6 \cdot Step5 \cdot Step4 \cdot Step3 \cdot Step2 \cdot Step1 \cdot \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Psi_{final} = \begin{pmatrix} 0.408 \\ 0.289 \\ 0.408 \\ 0.289 \\ 0.408 \\ 0.289 \\ 0.408 \\ 0.289 \end{pmatrix} \qquad \sqrt{\frac{1}{6}} = 0.408 \qquad \sqrt{\frac{1}{12}} = 0.289$$

According to this result $|\Psi\rangle$ has indeed been teleported to the bottom wire on the right, so the goal has been achieved. However, Brassard suggests that measurements on the top wires in the $|0\rangle$-$|1\rangle$ basis are also instructive. There are four possible measurement outcomes: $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. The projection operators for $|0\rangle$ and $|1\rangle$ are as follows.

Projection operator for $|0\rangle$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Projection operator for $|1\rangle$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

These calculations also show successful teleportation.

Measurement result for qubits x and y

Final 3-qubit state

$$2\cdot \mathrm{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathrm{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, I\right]\right]\cdot \Psi_{final} = \begin{pmatrix}0.816\\0.577\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$\begin{bmatrix}x & y & z\\ \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0.816\\0.577\end{pmatrix}\end{bmatrix}$$

$$2\cdot \mathrm{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathrm{kronecker}\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, I\right]\right]\cdot \Psi_{final} = \begin{pmatrix}0\\0\\0.816\\0.577\\0\\0\\0\\0\end{pmatrix}$$

$$\begin{bmatrix}x & y & z\\ \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0.816\\0.577\end{pmatrix}\end{bmatrix}$$

$$2\cdot \mathrm{kronecker}\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \mathrm{kronecker}\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, I\right]\right]\cdot \Psi_{final} = \begin{pmatrix}0\\0\\0\\0\\0.816\\0.577\\0\\0\end{pmatrix}$$

$$\begin{bmatrix}x & y & z\\ \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0.816\\0.577\end{pmatrix}\end{bmatrix}$$

$$2\cdot \mathrm{kronecker}\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \mathrm{kronecker}\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, I\right]\right]\cdot \Psi_{final} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0.816\\0.577\end{pmatrix}$$

$$\begin{bmatrix}x & y & z\\ \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}0.816\\0.577\end{pmatrix}\end{bmatrix}$$

Appendix

In the inverse controlled-NOT gate of steps 6 and 8 of Brassard's teleportation circuit, **c** is the control, **a** is the target and **b** is unchanged.

a	b	c	'	a'	b'	c'
0	0	0	'	0	0	0
0	0	1	'	1	0	1
0	1	0	'	0	1	0
0	1	1	'	1	1	1
1	0	0	'	1	0	0
1	0	1	'	0	0	1
1	1	0	'	1	1	0
1	1	1	'	0	1	1