Teleportation of Two Qubits

Frank Rioux

Emeritus Professor of Chemistry
College of St. Benedict | St. John's University

The following quantum circuit teleports |a>|b> from the top two wires on the left to the bottom two wires on the right.

Input	1	2	3	4	5	6		Output
$\vert a\rangle$	⋯	⋯	⋯	•	H	⋯	▷	Measure, 0 or 1
$\vert b\rangle$	⋯	⋯	⋯	|	•	H	▷	Measure, 0 or 1
$\vert 0\rangle$	H	•	⋯	⊕	|	⋯	▷	Measure, 0 or 1
$\vert 0\rangle$	H	|	•	⋯	⊕	⋯	▷	Measure, 0 or 1
$\vert 0\rangle$	⋯	⊕	|	⋯	⋯	⋯		$\vert a\rangle$
$\vert 0\rangle$	⋯	⋯	⊕	⋯	⋯	⋯		$\vert b\rangle$

There are 16 possible measurement results on the top four wires on the right ranging from |0> to |15> (|000000> to |111111> in binary notation), and these results determine the action required to transform the bottom two wires to |a>|b>.

As an initial example, if the teleportee in decimal notation is |3> the total input state is |48>.

Teleportee: $\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ = |3>   Total input state: $\Psi = \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}$ = |48>

Ψ_{48} is the only non-zero element in the input state vector.

$i := 0..63$ $\Psi_i := 0$ $\Psi_{48} := 1$ $\Psi^T =$

	44	45	46	47	48	49	50	51	52	53
0	0	0	0	0	1	0	0	0	0	...

With the assistance of truth tables provided in the Appendix and a bit of tedious algebra the operation of the teleportation circuit on this input state can be summarized as follows.

Given $|\Psi\rangle_{input} = |1\rangle|1\rangle|0\rangle|0\rangle|0\rangle|0\rangle = |48\rangle$ the teleportation circuit yields

$$|\Psi\rangle_{output} = \frac{1}{4}\left[\begin{matrix}(|3\rangle - |7\rangle - |11\rangle + |15\rangle)|00\rangle + (|2\rangle - |6\rangle - |10\rangle + |14\rangle)|01\rangle \\ +(|1\rangle - |5\rangle - |9\rangle + |13\rangle)|10\rangle + (|0\rangle - |4\rangle - |8\rangle + |12\rangle)|11\rangle\end{matrix}\right]$$

The output state shows that if the measurement result is 3, 7, 11 or 15, the NOT operator (|0> to |1> and |1> to |0>) should be applied to wires **5** and **6.** If the result is 2, 6, 10 or 14, the NOT operator is applied to wire **5**; if the result 1, 5, 9 or 13, the NOT operator is applied to wire **6**; if the result is 0, 4, 8 or 12, no action is required.

Now an example teleportation calculation will be carried out on the teleportee state given above. The following matrix operators are required for the teleportation circuit. They are the identity, the Hadamard gate, the measurement operators for |0> and |1>, and the CnNOT gate.

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad CnNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix operators for the six steps in the circuit are formed using tensor (Kronecker) multiplication.

$$Step1 := kronecker(I, kronecker(I, kronecker(H, kronecker(H, kronecker(I, I)))))$$

$$Step2 := kronecker(I, kronecker(I, kronecker(CnNOT, I))) \qquad Step3 := kronecker(I, kronecker(I, kronecker(I, CnNOT)))$$

$$Step4 := kronecker(CnNOT, kronecker(I, kronecker(I, I))) \qquad Step5 := kronecker(H, kronecker(CnNOT, kronecker(I, I)))$$

$$Step6 := kronecker(I, kronecker(H, kronecker(I, kronecker(I, kronecker(I, I)))))$$

$$Step7 := kronecker\left(M_0, kronecker\left(M_0, kronecker\left(M_0, kronecker\left(M_0, kronecker(I, I)\right)\right)\right)\right)$$

The output vector is calculated for the simple case of measuring |0>|0>|0>|0> on the top four wires. The factor of 4 takes into account that measuring |0>|0>|0>|0> has a probability amplitude of 1/4. It therefore normalizes the calculation.

$$\Psi_{out} := 4 \cdot Step7 \cdot Step6 \cdot Step5 \cdot Step4 \cdot Step3 \cdot Step2 \cdot Step1 \cdot \Psi$$

$\Psi_{out}^{T} =$

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	1	0	0	0	0	0	...

$$\Psi_{out} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ = |3>

In a second example a superposition of |0> + |1> + |2> + |3> is teleported.

Teleportee: $$\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ = (|0> + |1> + |2> + |3>)/2

Total input vector: $$\Psi = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Non-zero vector elements: $\Psi_0 := \frac{1}{2}$ $\quad \Psi_{16} := \frac{1}{2}$ $\quad \Psi_{32} := \frac{1}{2}$ $\quad \Psi_{48} := \frac{1}{2}$

The output vector is calculated for the case of measuring |0>|0>|0>|0> on the top four wires.

$$\Psi_{out} := 4 \cdot Step7 \cdot Step6 \cdot Step5 \cdot Step4 \cdot Step3 \cdot Step2 \cdot Step1 \cdot \Psi$$

$\Psi_{out}^{T} =$

	0	1	2	3	4	5	6	7	8	9
0	0.5	0.5	0.5	0.5	0	0	0	0	0	...

$$\Psi_{out} = \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{2}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

= (|0> + |1> + |2> + |3>)/2

A final example:

Teleportee: $\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}\sqrt{\frac{1}{3}}\\\sqrt{\frac{2}{3}}\end{pmatrix}$

Total input vector: $\Psi = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}\sqrt{\frac{1}{3}}\\\sqrt{\frac{2}{3}}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}$

Non-zero vector elements: $\Psi_0 := \sqrt{\frac{1}{6}}$ $\quad \Psi_{16} := \sqrt{\frac{1}{3}}$ $\quad \Psi_{32} := \sqrt{\frac{1}{6}}$ $\quad \Psi_{48} := \sqrt{\frac{1}{3}}$

$\Psi_{out} := 4\cdot Step7\cdot Step6\cdot Step5\cdot Step4\cdot Step3\cdot Step2\cdot Step1\cdot\Psi$

$\Psi_{out}^{T} =$

	0	1	2	3	4	5	6	7	8
0	0.408	0.577	0.408	0.577	0	0	0	0	...

$$\Psi_{out} = \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}\sqrt{\frac{1}{3}}\\\sqrt{\frac{2}{3}}\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0.408\\0.577\\0.408\\0.577\end{pmatrix}$$

Appendix

$$NOT = \begin{pmatrix}0 & \text{to} & 1\\1 & \text{to} & 0\end{pmatrix} \qquad H = \begin{bmatrix}0 & \text{to} & \frac{1}{\sqrt{2}}\cdot(0+1)\\1 & \text{to} & \frac{1}{\sqrt{2}}\cdot(0-1)\end{bmatrix}$$

$$CnNOT = \begin{pmatrix}\text{Decimal} & \text{Binary} & \text{to} & \text{Binary} & \text{Decimal}\\0 & 000 & \text{to} & 000 & 0\\1 & 001 & \text{to} & 001 & 1\\2 & 010 & \text{to} & 010 & 2\\3 & 011 & \text{to} & 011 & 3\\4 & 100 & \text{to} & 101 & 5\\5 & 101 & \text{to} & 100 & 4\\6 & 110 & \text{to} & 111 & 7\\7 & 111 & \text{to} & 110 & 6\end{pmatrix}$$