Superdense Coding

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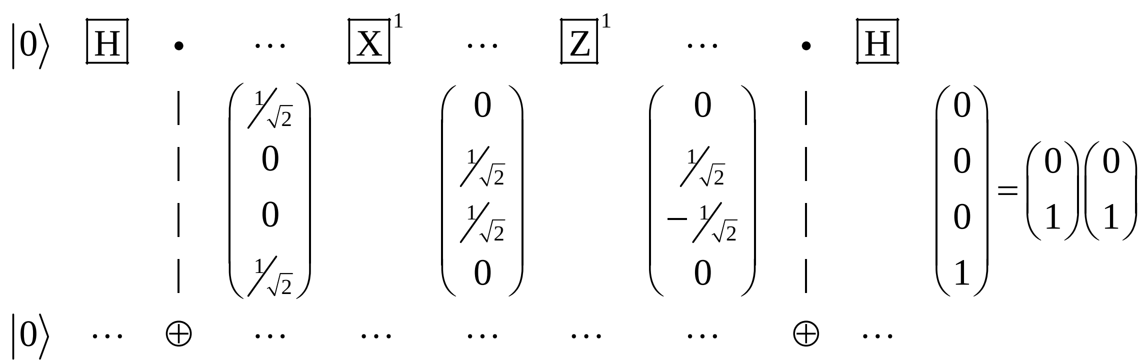
Quantum superdense coding reliably transmits two classical bits through an entangled pair of particles, even though only one member of the pair is handled by the sender. Charles Bennett, *Physics Today*, October 1995, p. 27

This tutorial is based on Brad Rubin's "Superdense Coding" at the Wolfram Demonstration Project: http:demonstrations.wolfram.com/SuperdenseCoding/. The quantum circuit shown below impliments quantum dense coding. Alice and Bob share the entangled pair of photons in the Bell basis shown at the left. Alice encodes two classical bits of information (four possible messages) on her photon, and Bob subsequently reads her message by performing a Bell state measurement on the modified entangled photon pair. In other words, although Alice encodes two bits on her photon Bob's readout requires a measurement involving both photons. In this example Alice sends |11> to Bob.



As shown above Alice and Bob share the following maximally entangled two-qubit state. It is easily recognized as one of four two-quibit Bell states.

$$\left|\Phi_p\right\rangle = \frac{1}{\sqrt{2}}\left[\left|0\right\rangle_A\left|0\right\rangle_B + \left|1\right\rangle_A\left|1\right\rangle_B\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \qquad \Phi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

The other Bell states are: $\Phi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \quad \Psi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \quad \Psi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$

Implementation of the superdense coding circuit requires the following matrix operators. Note on the right below that X^0 and Z^0 are the identity matrix (do nothing).

$$I := \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad H := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1&1\\1&-1\end{pmatrix} \quad CNOT := \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \quad X := \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad Z := \begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad X^0 = \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad Z^0 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

The CNOT followed by a Hadamard matrix on the top wire is a Bell state measurement, yielding the index of the Bell state produced by the X-Z operations on the top wire.

Input | Output

$B2 := 0 \quad B1 := 0 \qquad \text{kronecker}(H,I)\cdot CNOT\cdot \text{kronecker}\left(Z^{B2},I\right)\cdot \text{kronecker}\left(X^{B1},I\right)\cdot \Phi_p = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad B2\cdot B1 = \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$B2 := 0 \quad B1 := 1 \qquad \text{kronecker}(H,I)\cdot CNOT\cdot \text{kronecker}\left(Z^{B2},I\right)\cdot \text{kronecker}\left(X^{B1},I\right)\cdot \Phi_p = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad B2\cdot B1 = \begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$B2 := 1 \quad B1 := 0 \qquad \text{kronecker}(H,I)\cdot CNOT\cdot \text{kronecker}\left(Z^{B2},I\right)\cdot \text{kronecker}\left(X^{B1},I\right)\cdot \Phi_p = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad B2\cdot B1 = \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$B2 := 1 \quad B1 := 1 \qquad \text{kronecker}(H,I)\cdot CNOT\cdot \text{kronecker}\left(Z^{B2},I\right)\cdot \text{kronecker}\left(X^{B1},I\right)\cdot \Phi_p = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad B2\cdot B1 = \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

We finish with an algebraic derivation of the last result.

$$\frac{1}{\sqrt{2}}\left[|0\rangle|0\rangle + |1\rangle|1\rangle\right]$$

$$X \otimes I$$

$$\frac{1}{\sqrt{2}}\left[|1\rangle|0\rangle + |0\rangle|1\rangle\right]$$

$$Z \otimes I$$

$$\frac{1}{\sqrt{2}}\left[|0\rangle|1\rangle - |1\rangle|0\rangle\right]$$

$$CNOT$$

$$\frac{1}{\sqrt{2}}\left[|0\rangle|1\rangle - |1\rangle|1\rangle\right]$$

$$H \otimes I$$

$$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|1\rangle - \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|1\rangle\right] = |1\rangle|1\rangle = |3\rangle$$