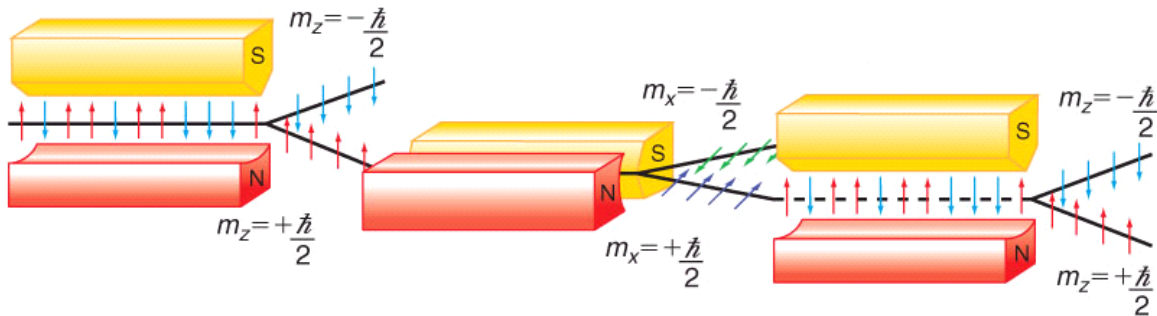


# Analysis of the Stern-Gerlach Experiment

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Silver atoms are deflected by an inhomogeneous magnetic field because of the two-valued magnetic moment associated with their unpaired 5s electron ([Kr]5s<sup>1</sup>4d<sup>10</sup>). The beam of silver atoms entering the Stern-Gerlach magnet oriented in the z-direction (SGZ) on the left is unpolarized. This means it is a mixture of randomly polarized Ag atoms. A mixture cannot be represented by a wave function, it requires a density matrix, as will be shown later.

This situation is exactly analogous to the three-polarizer demonstration. Light emerging from an incandescent light bulb is unpolarized, a mixture of all possible polarization angles, so we can't write a wave function for it. The first Stern-Gerlach magnet plays the same role as the first polarizer, it forces the Ag atoms into one of measurement eigenstates - spin-up or spin-down in the z-direction. The only difference is that in the three-polarizer demonstration only one state was created - vertical polarization. Both demonstrations illustrate that the only values that are observed in an experiment are the eigenvalues of the measurement operator.

To continue with the analysis of the Stern-Gerlach demonstration we need vectors to represent the various spin states of the Ag atoms.

## Spin Eigenfunctions

Spin-up in the z-direction:  $\alpha_z := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$       Spin-down in the z-direction:  $\beta_z := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Spin-up in the x-direction:  $\alpha_x := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$       Spin-down in the x-direction:  $\beta_x := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

In the next step, the spin-up beam (deflected toward by the magnet's north pole) enters a magnet oriented in the x-direction, SGX. The  $\alpha_z$  beam splits into  $\alpha_x$  and  $\beta_x$  beams of equal intensity. This is because it is a superposition of the x-direction spin eigenstates as shown below.

$$\frac{1}{\sqrt{2}} \cdot \left[ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot (\alpha_x + \beta_x) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Next the  $\alpha_x$  beam is directed toward a second SGZ magnet and splits into two equal  $\alpha_z$  and  $\beta_z$  beams. This happens because  $\alpha_x$  is a superposition of the  $\alpha_z$  and  $\beta_z$  spin states.

$$\frac{1}{\sqrt{2}} \cdot \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot (\alpha_z + \beta_z) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

## Operators

We can also use the Pauli operators (in units of  $\hbar/4\pi$ ) to analyze this experiment.

$$\text{SGZ operator:} \quad \text{SGZ} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{SGX operator:} \quad \text{SGX} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The probability that an  $\alpha_z$  Ag atom will emerge spin-up after passing through a SGX magnet:

$$\text{Probability amplitude:} \quad \alpha_x^T \cdot \text{SGX} \cdot \alpha_z = 0.707 \quad \text{Probability:} \quad \left( \alpha_x^T \cdot \text{SGX} \cdot \alpha_z \right)^2 = 0.5$$

The probability that an  $\alpha_z$  Ag atom will emerge spin-down after passing through a SGX magnet:

$$\text{Probability amplitude:} \quad \beta_x^T \cdot \text{SGX} \cdot \alpha_z = -0.707 \quad \text{Probability:} \quad \left( \beta_x^T \cdot \text{SGX} \cdot \alpha_z \right)^2 = 0.5$$

The probability that an  $\alpha_x$  Ag atom will emerge spin-up after passing through a SGZ magnet:

$$\text{Probability amplitude:} \quad \alpha_z^T \cdot \text{SGX} \cdot \alpha_x = 0.707 \quad \text{Probability:} \quad \left( \alpha_z^T \cdot \text{SGX} \cdot \alpha_x \right)^2 = 0.5$$

The probability that an  $\alpha_x$  Ag atom will emerge spin-down after passing through a SGZ magnet:

$$\text{Probability amplitude:} \quad \beta_z^T \cdot \text{SGX} \cdot \alpha_x = 0.707 \quad \text{Probability:} \quad \left( \beta_z^T \cdot \text{SGX} \cdot \alpha_x \right)^2 = 0.5$$

In examining the figure above we note that the SGX magnet destroys the entering  $\alpha_z$  state, creating a superposition of spin-up and spin-down in the x-direction. Again measurement forces the system into one of the eigenstates of the measurement operator.

## Density Operator (Matrix) Approach

A more general analysis is based on the concept of the density operator (matrix), in general given by the following outer product  $|\Psi\rangle\langle\Psi|$ . It is especially important because it can be used to represent mixtures, which cannot be represented by wave functions as noted above.

For example, the probability that an  $\alpha_z$  spin system will emerge in the  $\alpha_x$  channel of a SGX magnet is equal to the trace of the product of the density matrices representing the  $\alpha_z$  and  $\alpha_x$  states as shown below.

$$\begin{aligned} \left| \langle \alpha_x | \alpha_z \rangle \right|^2 &= \langle \alpha_z | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle = \sum_i \langle \alpha_z | i \rangle \langle i | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle = \sum_i \langle i | \alpha_x \rangle \langle \alpha_x | \alpha_z \rangle \langle \alpha_z | i \rangle \\ &= \text{Tr} \left( |\alpha_x\rangle\langle\alpha_x| \alpha_z \right) = \text{Tr} \left( \widehat{\rho}_{\alpha_x} \widehat{\rho}_{\alpha_z} \right) \end{aligned}$$

where the completeness relation  $\sum_i |i\rangle\langle i| = 1$  has been employed.

Density matrices for spin-up and spin-down in the z-direction:

$$\rho_{\alpha z} := \alpha_z \cdot \alpha_z^T$$

$$\rho_{\beta z} := \beta_z \cdot \beta_z^T$$

Density matrices for spin-up and spin-down in the x-direction:

$$\rho_{\alpha x} := \alpha_x \cdot \alpha_x^T$$

$$\rho_{\beta x} := \beta_x \cdot \beta_x^T$$

An unpolarized spin system can be represented by a 50-50 mixture of any two orthogonal spin density matrices. Below it is shown that using the z-direction and the x-direction give the same answer.

$$\rho_{\text{mix}} := \frac{1}{2} \cdot \rho_{\alpha z} + \frac{1}{2} \cdot \rho_{\beta z} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

Now we re-analyze the Stern-Gerlach experiment using the density operator (matrix) approach.

The probability that an unpolarized spin system will emerge in the  $\alpha_z$  channel of a SGZ magnet is 0.5:

$$\text{tr}(\rho_{\alpha z} \cdot \rho_{\text{mix}}) = 0.5$$

The probability that the  $\alpha_z$  beam will emerge in the  $\alpha_x$  channel of a SGX magnet is 0.5:

$$\text{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) = 0.5$$

The probability that the  $\alpha_x$  beam will emerge in the  $\alpha_z$  channel of the final SGZ magnet is 0.5:

$$\text{tr}(\rho_{\alpha z} \cdot \rho_{\alpha x}) = 0.5$$

The probability that the  $\alpha_x$  beam will emerge in the  $\beta_z$  channel of the final SGZ magnet is 0.5:

$$\text{tr}(\rho_{\beta z} \cdot \rho_{\alpha x}) = 0.5$$

After the final SGZ magnet, 1/8 of the original Ag atoms emerge in the  $\alpha_z$  channel and 1/8 in the  $\beta_z$  channel.

$$\text{tr}(\rho_{\alpha z} \cdot \rho_{\alpha x}) \cdot \text{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) \cdot \text{tr}(\rho_{\alpha z} \cdot \rho_{\text{mix}}) = 0.125 \quad \text{tr}(\rho_{\beta z} \cdot \rho_{\alpha x}) \cdot \text{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) \cdot \text{tr}(\rho_{\alpha z} \cdot \rho_{\text{mix}}) = 0.125$$