

A Simple Teleportation Circuit

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This circuit teleports $|\Psi\rangle$ from the top wire to the bottom wire. Measurement on the top wire yields $|0\rangle$ or $|1\rangle$, the power to which the Z matrix on the bottom wire is raised. As shown below Z^0 is the identity matrix, in other words do nothing because $|\Psi\rangle$ is already on the bottom wire.

$$\begin{array}{c}
 |\Psi\rangle \triangleright \bullet \text{ H} \triangleright \text{Measure } m = 0 \text{ or } 1 \\
 | \quad | \\
 |0\rangle \triangleright \oplus \triangleright Z^m \rightarrow |\Psi\rangle
 \end{array}
 \quad
 \begin{array}{l}
 |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |\Psi\rangle = \begin{pmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{pmatrix}
 \end{array}$$

The required quantum gates in matrix format:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Z^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Form the quantum circuit for teleportation:

$$\text{QC} := \text{kroncker}(H, I) \cdot \text{CNOT} \quad \text{QC} = \begin{pmatrix} 0.707 & 0 & 0 & 0.707 \\ 0 & 0.707 & 0.707 & 0 \\ 0.707 & 0 & 0 & -0.707 \\ 0 & 0.707 & -0.707 & 0 \end{pmatrix} \quad \text{QC} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Write the initial state as a 4-vector using tensor multiplication.

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

Calculate the output state of the circuit and write it as a superposition of $|0\rangle$ and $|1\rangle$ on the top wire.

$$\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \right]$$

It is clear that if $|0\rangle$ is measured on the top wire, $|\Psi\rangle$ is on the bottom wire without further action. If $|1\rangle$ is measured on the top wire the qubit on the bottom wire is converted to $|\Psi\rangle$ by multiplication by Z.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

Matrix summary:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \right]$$

Algebraic summary:

$$Z = \begin{pmatrix} 0 & \text{to} & 0 \\ 1 & \text{to} & -1 \end{pmatrix} \quad H = \begin{bmatrix} 0 & \text{to} & \frac{(0+1)}{\sqrt{2}} \\ 1 & \text{to} & \frac{(0-1)}{\sqrt{2}} \end{bmatrix} \quad \text{CNOT} = \begin{pmatrix} 00 & \text{to} & 00 \\ 01 & \text{to} & 01 \\ 10 & \text{to} & 11 \\ 11 & \text{to} & 10 \end{pmatrix}$$

$$(\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle)|0\rangle = \sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|10\rangle$$

CNOT

$$\sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$$

H ⊗ I

$$\frac{1}{\sqrt{2}} \left[\sqrt{\frac{1}{3}}(|0\rangle + |1\rangle)|0\rangle + \sqrt{\frac{2}{3}}(|0\rangle - |1\rangle)|1\rangle \right] = \frac{1}{\sqrt{2}} \left[|0\rangle(\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle) + |1\rangle(\sqrt{\frac{1}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle) \right]$$

↓

$$\frac{1}{\sqrt{2}} \left[|0\rangle \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} + |1\rangle \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \right] \xrightarrow{\text{Action}} \frac{1}{\sqrt{2}} \left[I \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} + Z \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \right]$$

Total matrix operator approach:

$$m = 0 \quad \text{kronecker}(I, Z^0) \cdot \text{kronecker} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, I \right] \cdot \text{kronecker}(H, I) \cdot \text{CNOT} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0.408 \\ 0.577 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$\mathbf{m = 1} \quad \text{kroncker}(I, Z^1) \cdot \text{kroncker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, I\right] \cdot \text{kroncker}(H, I) \cdot \text{CNOT} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.408 \\ 0.577 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

The measurement projection operators used above in the next to the last step are:

$$\mathbf{m = 0} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1 \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{m = 1} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (0 \ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$ \Psi\rangle$	\triangleright	\bullet	H	$ m\rangle\langle m $	\bullet	\triangleright	$ m\rangle$
$ 0\rangle$	\triangleright	\oplus			\boxed{Z}	\triangleright	$ \Psi\rangle$