Matrix Math for "Qubit Quantum Mechanics..." by Enrique Galvez, AJP 78, 510-519 (2010)

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In this document I reproduce most of the results presented in Professor Galvez's paper using the Mathcad programming environment.

State Vectors

Photon moving horizontally: $x := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Photon moving vertically: $y := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Null vector: $n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Horizontal polarization: $h := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Vertical polarization: $v := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Diagonal polarization: $d := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Single mode operators:

Projection operators for motion in the x- and y-directions: $X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Operator for polarizing film oriented at angle of θ to the horizontal. $\Theta_{op}(\theta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\cdot\sin(\theta) \\ \cos(\theta)\cdot\sin(\theta) & \sin(\theta)^2 \end{pmatrix}$

Beam splitter: $BS := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ Mirror: $M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Phase shift: $A(\delta) := \begin{pmatrix} e^{i\cdot\delta} & 0 \\ 0 & 1 \end{pmatrix}$

Half and quarter wave plate: $W_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $W_4 := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ Identity: $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotated half wave plate: $W_2(\theta) := \begin{pmatrix} \cos(2\cdot\theta) & \sin(2\cdot\theta) \\ \sin(2\cdot\theta) & -\cos(2\cdot\theta) \end{pmatrix}$ $W(\theta) := \begin{pmatrix} \cos(2\cdot\theta) & \sin(2\cdot\theta) & 0 & 0 \\ \sin(2\cdot\theta) & -\cos(2\cdot\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Mach-Zehnder interferometer: $MZ(\delta) := BS\cdot A(\delta)\cdot M\cdot BS$

Two mode states and operators:

Single-photon direction of propagation and polarization states:

$xh := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $xv := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $yh := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $yv := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Two-photon direction of propagation states.

$$xx := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad xy := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad yx := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad yy := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Two-photon polarization states.

$$hh := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad hv := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad vh := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad vv := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Polarizing beam splitter which transmits horizontally polarized photons and reflects vertically polarized photons.

$$PBS := xh \cdot xh^T + yv \cdot xv^T + yh \cdot yh^T + xv \cdot yv^T \qquad PBS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Polarization M-Z interferometer: $MZ_P(\delta) := PBS \cdot kronecker(A(\delta), I) \cdot kronecker(M, I) \cdot PBS$

Kronecker is Mathcad's command for tensor multiplication of square matrices.

Mach-Zehnder interferometer for direction of propagation and polarization, which places a rotatable half-wave plate in the upper path.

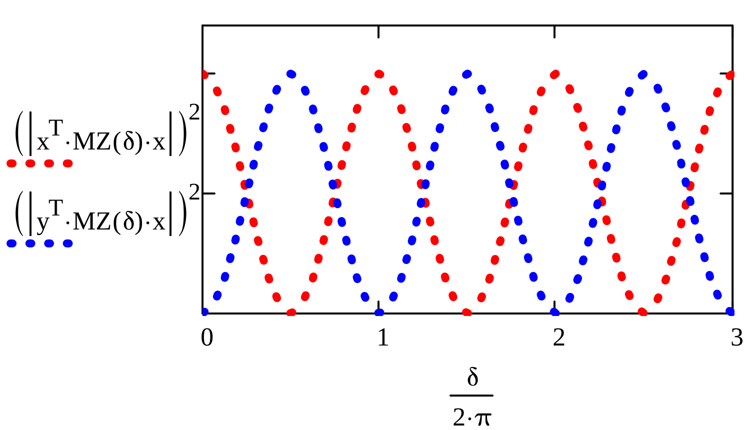
$$MZ_{dp}(\theta, \delta) := kronecker(BS, I) \cdot kronecker(A(\delta), I) \cdot W(\theta) \cdot kronecker(M, I) \cdot kronecker(BS, I)$$

Mach-Zehnder two-photon direction-of-propagation interferometer.

$$BSBS := kronecker(BS, BS) \qquad MM := kronecker(M, M) \qquad AA(\delta) := kronecker(A(\delta), A(\delta))$$

$$MZ_{dd}(\delta) := BSBS \cdot AA(\delta) \cdot MM \cdot BSBS$$

Confirm the results in Figure 2 for the Mach-Zehnder interferometer: $\delta := 0, .125 \cdot \pi .. 6\pi$



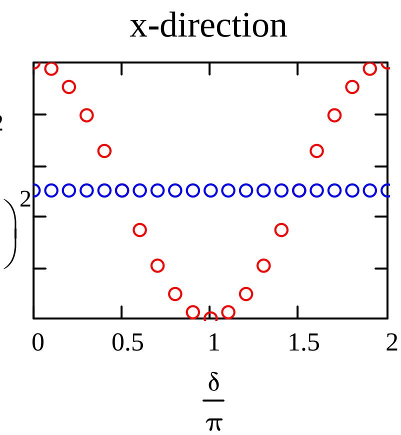
Demonstrate that a superposition is formed after first beam splitter

$$BS \cdot x = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (x + i \cdot y) = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$$

Confirmation that path information destroys interference. $\delta := 0, .1 \cdot \pi .. 2 \cdot \pi$

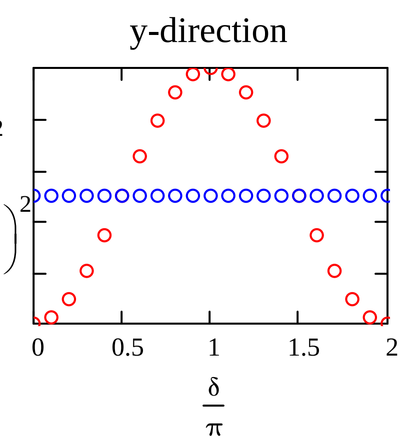
θ = 0, no path information $\left(\left|\text{kronecker}(X, I) \cdot MZ_{dp}(0, \delta) \cdot xv\right|\right)^2$

θ = π/4, path information $\left(\left|\text{kronecker}(X, I) \cdot MZ_{dp}\left(\frac{\pi}{4}, \delta\right) \cdot xv\right|\right)^2$



θ = 0, no path information $\left(\left|\text{kronecker}(Y, I) \cdot MZ_{dp}(0, \delta) \cdot xv\right|\right)^2$

θ = π/4, path information $\left(\left|\text{kronecker}(Y, I) \cdot MZ_{dp}\left(\frac{\pi}{4}, \delta\right) \cdot xv\right|\right)^2$



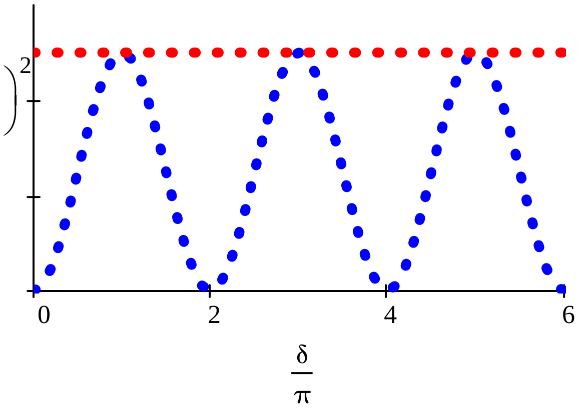
Erasure of path information restores interference. Erasers for the x- and y-directions place diagonal polarizers in those directions after the interferometer.

$$E_x := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad E_y := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The x-direction has an eraser and the y-direction does not. $\delta := 0, .1\cdot\pi .. 6\pi$

x-direction: $\left(\left|\text{kronecker}(X,I)\cdot E_x\cdot MZ_{dp}\left(\frac{\pi}{4},\delta\right)\cdot xv\right|\right)^2$

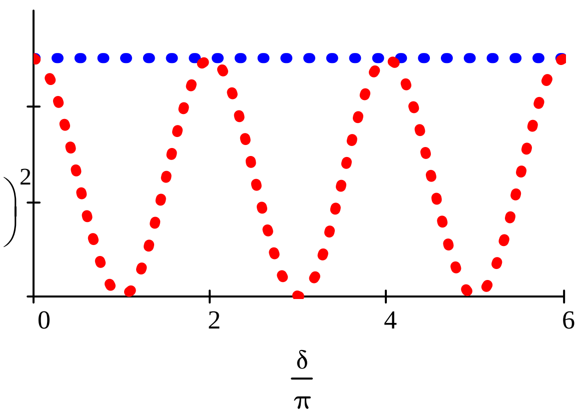
y-direction: $\left(\left|\text{kronecker}(Y,I)\cdot MZ_{dp}\left(\frac{\pi}{4},\delta\right)\cdot xv\right|\right)^2$



The y-direction has an eraser and the x-direction does not.

x-direction: $\left(\left|\text{kronecker}(X,I)\cdot MZ_{dp}\left(\frac{\pi}{4},\delta\right)\cdot xv\right|\right)^2$

y-direction: $\left(\left|\text{kronecker}(Y,I)\cdot E_y\cdot MZ_{dp}\left(\frac{\pi}{4},\delta\right)\cdot xv\right|\right)^2$



For the MZ polarization interferometer diagonally polarized light enters in the x-direction, |xd >. Tensor vector multiplication is awkward in Mathcad as is shown below.

$$\Psi_{in} := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix} \qquad \text{submatrix}(\text{kronecker}(\text{augment}(x,n),\text{augment}(d,n)),1,4,1,1) = \begin{pmatrix}0.707\\0.707\\0\\0\end{pmatrix}$$

No light, however, exits in the x-direction. It exits in the y-direction showing no interference effects.

$\delta := 0, .2\cdot\pi .. \pi$

x-direction: $\left(\left|\text{kronecker}(X,I)\cdot MZ_P(\delta)\cdot\Psi_{in}\right|\right)^2 =$

0
0
0
0
0
0

y-direction: $\left(\left|\text{kronecker}(Y,I)\cdot MZ_P(\delta)\cdot\Psi_{in}\right|\right)^2 =$

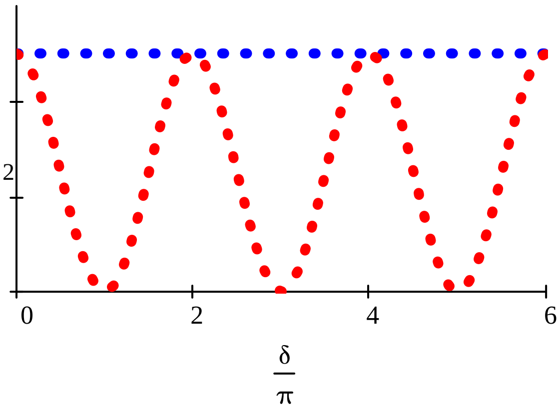
1
1
1
1
1
1

Placement of a **D** polarizer in the y-direction output erases distinguishing information and interference appears.

$\delta := 0, .1\cdot\pi .. 6\pi$

x-direction: $\left(\left|\text{kronecker}(X,I)\cdot MZ_{dp}\left(\frac{\pi}{4},\delta\right)\cdot xv\right|\right)^2$

y-direction: $\left(\left|\text{kronecker}(Y,I)\cdot E_y\cdot MZ_{dp}\left(\frac{\pi}{4},\delta\right)\cdot xv\right|\right)^2$



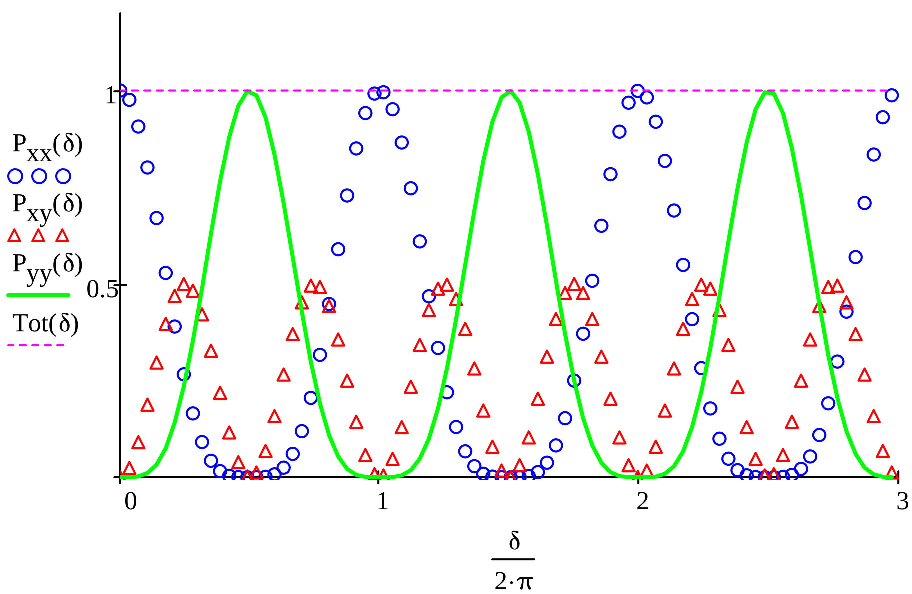
Calculation of exit probabilities for two photons in direction-of-propagation modes:

$$P_{xx}(\delta) := \left(\left|xx^T\cdot MZ_{dd}(\delta)\cdot xx\right|\right)^2 \qquad P_{xy}(\delta) := \left[\left|\frac{1}{\sqrt{2}}\cdot(xy + yx)^T\cdot MZ_{dd}(\delta)\cdot xx\right|\right]^2$$

$$P_{yy}(\delta) := \left(\left|yy^T\cdot MZ_{dd}(\delta)\cdot xx\right|\right)^2 \qquad Tot(\delta) := P_{xx}(\delta) + P_{xy}(\delta) + P_{yy}(\delta)$$

Reproduction of Figure 5b with the addition of P_{yy}.

$\delta := 0, .07\cdot\pi .. 6\cdot\pi$



"The striking result is that the (P_{xy}) interference pattern has twice the frequency of the single-photon interference pattern. Nonclassical interference shows new quantum aspects: two photons acting as a single quantum object (a biphoton)."

Hong-Ou-Mandel interference (right column, page 516):

$$BSBS\cdot\frac{1}{\sqrt{2}}\cdot(xy + yx) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix} \qquad \frac{i}{\sqrt{2}}\cdot(xx + yy) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix}$$

Section III.D deals with distinguishing between pure and mixed states experimentally. The pure state and it density matrix are given below.

$$\Psi_{pure} := \frac{1}{\sqrt{2}}\cdot(hh + vv) \qquad \Psi_{pure} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \qquad \Psi_{pure}\cdot\Psi_{pure}^{T} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

The density matrix for the mixed state is calculated as follows.

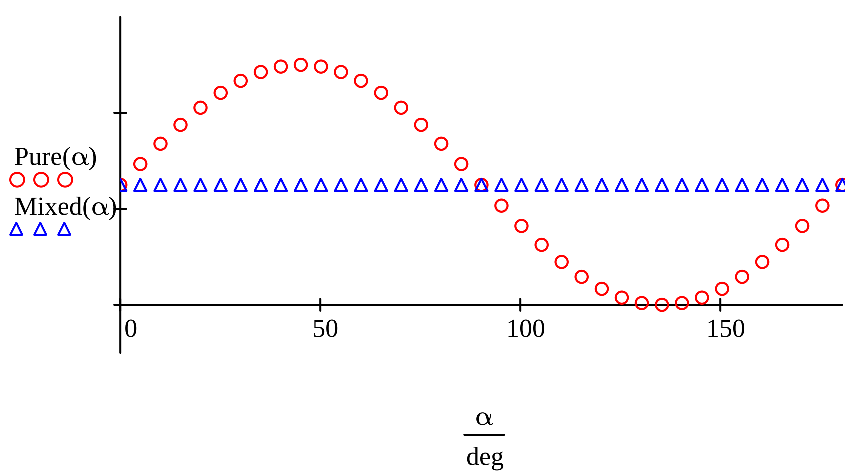
$$\frac{1}{2}\cdot hh\cdot hh^{T} + \frac{1}{2}\cdot vv\cdot vv^{T} = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

The following calculations and their graphical representation are in complete agreement with section III.D

$$Pure(\alpha) := tr\left[\frac{1}{2}\cdot\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\cdot\frac{1}{2}\cdot\left[\begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix}\cdot\begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix}^{T}\right]\right] \text{ simplify } \rightarrow \frac{\sin(2\cdot\alpha)}{4} + \frac{1}{4}$$

$$Mixed(\alpha) := tr\left[\frac{1}{2}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\frac{1}{2}\cdot\left[\begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix}\cdot\begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix}^{T}\right]\right] \text{ simplify } \rightarrow \frac{1}{4}$$

Reproduce Figure 6 results. $\alpha := 0\cdot deg, 5\cdot deg .. 180\cdot deg$



The following calculation are in agreement with the math in the final paragraph of section IV.D.

$$\mathrm{kronecker}\left(W_2(0), I\right)\cdot\Psi_{pure} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \qquad \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}\cdot\begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

$$\mathrm{kronecker}\left(W_2(0), I\right)\cdot\Psi_{pure}\cdot\Psi_{pure}^T\cdot\mathrm{kronecker}\left(W_2(0), I\right)^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

$$\mathrm{Pure}(\alpha) := \mathrm{tr}\left[\frac{1}{2}\cdot\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\cdot\frac{1}{2}\cdot\left[\begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix}\cdot\begin{pmatrix} \cos(\alpha) \\ \cos(\alpha) \\ \sin(\alpha) \\ \sin(\alpha) \end{pmatrix}^T\right]\right] \text{ simplify } \rightarrow \frac{1}{4} - \frac{\sin(2\cdot\alpha)}{4}$$

The paper shows this as [1 - sin(α)]/4 which I am confident is a typographical error.