Quantum Restrictions on Cloning

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Suppose a quantum copier exists which is able to carry out the following cloning operation.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{Clone}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Next the cloning operation (using the same copier) is carried out on the general qubit shown below.

$$\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \xrightarrow{\text{Clone}} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) \\ \cos(\theta)\sin(\theta) \\ \sin(\theta)\cos(\theta) \\ \sin^2(\theta) \end{pmatrix}$$

Quantum transformations are unitary, meaning probability is preserved. This requires that the scalar products of the initial and final states must be the same.

Initial state: $$\begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin(\theta)$$

Final state: $$\begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \sin^2(\theta)$$

It is clear from this analysis that quantum theory puts a significant restriction on copying. Only states for which sin(θ) = 0 or 1 (0 and 90 degrees) can be copied by the original cloner.

In conclusion, two quotes from Wootters and Zurek, *Physics Today*, February 2009, page 76.

> Perfect copying can be achieved only when the two states are orthogonal, and even then one can copy those two states (...) only with a copier specifically built for that set of states.
>
> In sum, one cannot make a perfect copy of an unknown quantum state, since, without prior knowledge, it is impossible to select the right copier for the job. That formulation is one common way of stating the no-cloning theorem.

An equivalent way to look at this (see arXiv:1701.00989v1) is to assume that a cloner exists for the V-H polarization states.

$$\hat{C}|V\rangle|X\rangle = |V\rangle|V\rangle \qquad \hat{C}|H\rangle|X\rangle = |H\rangle|H\rangle$$

A diagonally polarized photon is a superposition of the V-H polarization states.

$$|D\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)$$

However, due to the linearity of quantum mechanics the V-H cloner cannot clone a diagonally polarized photon.

$$\hat{C}|D\rangle|X\rangle = \hat{C}\frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)|X\rangle = \frac{1}{\sqrt{2}}\hat{C}\left(|V\rangle|X\rangle + |H\rangle|X\rangle\right) = \frac{1}{\sqrt{2}}\left(|V\rangle|V\rangle + |H\rangle|H\rangle\right)$$

$$\hat{C}|D\rangle|X\rangle \neq |D\rangle|D\rangle = \frac{1}{2}\left(|V\rangle|V\rangle + |V\rangle|H\rangle + |H\rangle|V\rangle + |H\rangle|H\rangle\right)$$