Solving Equations Using a Quantum Circuit

Frank Rioux
Professor Emeritus of Chemistry
CSB|SJU

This tutorial demonstrates the solution of two linear simultaneous equation using a quantum circuit. The circuit is taken from **arXiv:1302.1210**. See this reference for details on the experimental implementation of the circuit and also for a discussion of the potential of quantum solutions for systems of equations. Two other sources (**arXiv:1302.1946** and **1302.4310**) provide alternative quantum circuits and methods of implementation.

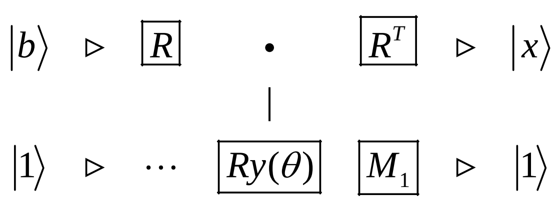
First we consider the conventional method of solving systems of linear equation for a particular matrix **A** and three different |b> vectors.

$$A|x\rangle = |b\rangle \qquad |x\rangle = A^{-1}|b\rangle$$

$$A := \begin{pmatrix} 1.5 & .5 \\ .5 & 1.5 \end{pmatrix} \qquad b_1 := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad b_2 := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad b_3 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A^{-1} \cdot b_1 = \begin{pmatrix} 0.354 \\ 0.354 \end{pmatrix} \qquad A^{-1} \cdot b_2 = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \qquad A^{-1} \cdot b_3 = \begin{pmatrix} 0.75 \\ -0.25 \end{pmatrix}$$

Next we show the quantum circuit (arXiv:1302.1210) that generates the same solutions. The **Appendix** considers two other equivalent circuits from this reference.



In this circuit, **R** is the matrix of eigenvectors of matrix **A** and **R**^{T} its transpose. The last step on the bottom wire is the measurement of |1>, which is represented by the projection operator **M**$_1$. The identity operator is required for cases in which a quantum gate operation is occurring on one wire and no operation is occurring on the other wire.

$$R := \text{eigenvecs}(A) \qquad R = \begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix} \qquad R^T = \begin{pmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{pmatrix} \qquad M_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ <-- } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The controlled rotation, CR(θ), is the only two-qubit gate in the circuit. The rotation angle required is determined by the ratio of the eigenvalues of **A** as shown below.

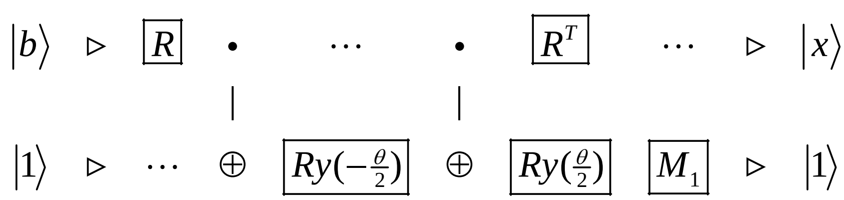
$$CR(\theta) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ 0 & 0 & \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad \text{eigenvals}(A) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \theta := -2 \cdot \text{acos}\left(\frac{1}{2}\right)$$

The input (|b>|1>) and output (|x>|1>) states are expressed in tensor format. **Kronecker** is Mathcad's command for the tensor product of matrices.

Input |b>|1>	Quantum Circuit	Output |x>|1>
$\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$\text{kronecker}\left(R^T, M_1\right)\cdot CR(\theta)\cdot\text{kronecker}(R, I)\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0.354\\0\\0.354\end{pmatrix}$	$\begin{pmatrix}0.354\\0.354\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$
$\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$	$\text{kronecker}\left(R^T, M_1\right)\cdot CR(\theta)\cdot\text{kronecker}(R, I)\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0.707\\0\\-0.707\end{pmatrix}$	$\begin{pmatrix}0.707\\-0.707\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\text{kronecker}\left(R^T, M_1\right)\cdot CR(\theta)\cdot\text{kronecker}(R, I)\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0.75\\0\\-0.25\end{pmatrix}$	$\begin{pmatrix}0.75\\-0.25\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$

Appendix

The alternative two-wire circuit shown in Fig. 1C requires CNOT and Ry rotation matrices.



$$\text{CNOT} := \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \qquad \text{Ry}(\theta) := \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

The quantum circuit is set up as follows. The input and output states are the same as the previous circuit.

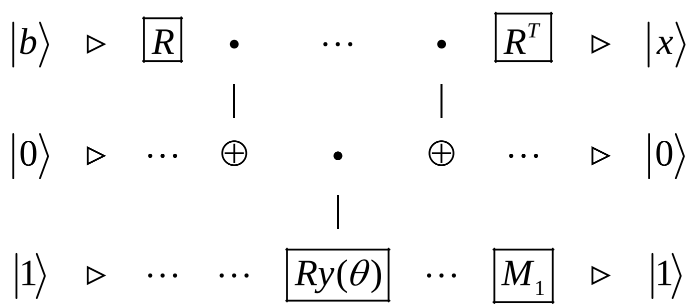
$$\text{QuantumCircuit} := \text{kronecker}\left(I, M_1\right)\cdot\text{kronecker}\left(R^T, \text{Ry}\left(\frac{\theta}{2}\right)\right)\cdot\text{CNOT}\cdot\text{kronecker}\left(I, \text{Ry}\left(\frac{-\theta}{2}\right)\right)\cdot\text{CNOT}\cdot\text{kronecker}(R, I)$$

Input |b>|1>	Quantum Circuit	Output |x>|1>
$\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$\text{QuantumCircuit}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0.354\\0\\0.354\end{pmatrix}$	$\begin{pmatrix}0.354\\0.354\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$

$$\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \qquad \text{QuantumCircuit}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0.707\\0\\-0.707\end{pmatrix} \qquad \begin{pmatrix}0.707\\-0.707\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \text{QuantumCircuit}\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0.75\\0\\-0.25\end{pmatrix} \qquad \begin{pmatrix}0.75\\-0.25\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

The three-wire circuit in Fig. 1B produces the following transformation: |b>|0>|1> --> |x>|0>|1>.



$$QC := \text{kronecker}\left(R^T, \text{kronecker}\left(I, M_1\right)\right)\cdot\text{kronecker}(\text{CNOT}, I)\cdot(\text{kronecker}(I, CR(\theta))\cdot\text{kronecker}(\text{CNOT}, I)\cdot\text{kronecker}(R, \text{kronecker}(I, I)))$$

Input |b>|0>|1> Quantum Circuit Output |x>|0>|1>

$$\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix} \qquad QC\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0.354\\0\\0\\0\\0.354\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0.354\\0.354\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\0\\0\\-1\\0\\0\end{pmatrix} \qquad QC\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\0\\0\\0\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0.707\\0\\0\\0\\-0.707\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0.707\\-0.707\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad QC \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.75 \\ 0 \\ 0 \\ 0 \\ -0.25 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0.75 \\ -0.25 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$