Single-photon Interference in a Mach-Zehnder Interferometer

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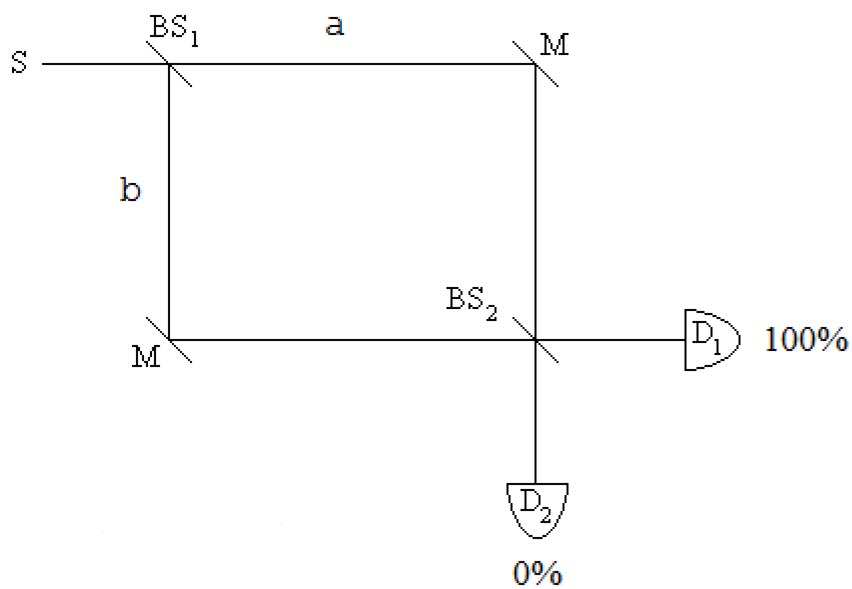
This analysis of the operation of a Mach-Zehnder Interferometer (MZI) will use tensor algebra and the creation and annihilation operators.



An interferometer arm can be occupied or unoccupied. These states are represented by the following vectors.

Unoccupied: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Occupied: $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

After the first beam splitter the photon is in an even superposition of being in both arms of the interferometer. By convention a 90 degree phase shift is assigned to arm **b** to preserve probability. In terms of the concept of occupation, the superposition takes the following form in tensor algebra.

$$|S\rangle \xrightarrow{BS1} \frac{1}{\sqrt{2}}\left[|1\rangle_a|0\rangle_b + i|0\rangle_a|1\rangle_b\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_a \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_b + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}_a \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_b\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}$$

The matrix operators required for this analysis are as follows.

Creation	Annihilation	Number	Identity
$C := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$N := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The effect of the creation, annihilation and number operators on |0> and |1>:

$$C \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad N \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad N \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The creation operator is the Hermitian adjoint of the annihilation operator and the annihilation operator is the Hermitian adjoint of the creation operator.

$$\overline{\left(A^T\right)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \overline{\left(C^T\right)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The number operator is the product of the creation and annihilation operators.

$$C \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \overline{\left(A^T\right)} \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad C \cdot \overline{\left(C^T\right)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvectors of the number operator are |0> and |1> with eigenvalues 0 and 1, respectively:

$$\text{eigenvecs}(N) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{eigenvals}(N) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

There are two paths to each detector. This provides the opportunity for constructive and destructive interference. To arrive at D_1 the **a-arm** photon state is reflected (90 degree phase shift) at BS_2 and the **b-arm** photon state is transmitted at BS_2. Therefore, photon detection requires the annihilation of the superposition of these paths to D_1. The annihilation is achieved with the following operator.

$$\frac{iA_a + A_b}{\sqrt{2}}$$

The product of this operator with its Hermitian conjugate (see above) creates the number operator for photon detection at D_1.

$$N_{D1} = \frac{-iC_a + C_b}{\sqrt{2}} \cdot \frac{iA_a + A_b}{\sqrt{2}}$$

The D_1 number operator is formed using Mathcad's kronecker command as follows.

$$N_{D1} := \frac{1}{2} \cdot (-i\,\text{kronecker}(C, I) + \text{kronecker}(I, C)) \cdot (i \cdot \text{kronecker}(A, I) + \text{kronecker}(I, A))$$

To arrive at D_2 the **a-arm** photon state is transmitted at BS_2 and the **b-arm** photon state is reflected (90 degree phase shift) at BS_2. Photon detection at D_2 requires the annihilation of the superposition of these paths to the detector. The annihilation is represented the following operator.

$$\frac{A_a + iA_b}{\sqrt{2}}$$

Therefore, the number operator for photon detection at D_2 is: $N_{D2} = \frac{C_a - iC_b}{\sqrt{2}} \cdot \frac{A_a + iA_b}{\sqrt{2}}$

The D_2 number operator is formed using Mathcad's kronecker command as follows.

$$N_{D2} := \frac{1}{2} \cdot (\text{kronecker}(C, I) - i \cdot \text{kronecker}(I, C)) \cdot (\text{kronecker}(A, I) + i \cdot \text{kronecker}(I, A))$$

We now show that the photon always arrives at D_1 and never at D_2 for an equal arm MZI.

Expectation value for photon detection at D_1: $\overline{(\Psi^T)} \cdot N_{D1} \cdot \Psi = 1$

Expectation value for photon detection at D_2: $\overline{(\Psi^T)} \cdot N_{D2} \cdot \Psi = 0$

Equivalent results can be obtained algebraically. Operating on Ψ with the D_1 number operator yields Ψ. In other words, Ψ is an eigenfunction of N_{D1} with eigenvalue 1.

$$\left[\frac{-iC_a + C_b}{\sqrt{2}} \cdot \frac{iA_a + A_b}{\sqrt{2}}\right] \frac{1}{\sqrt{2}}\left[|1\rangle_a |0\rangle_b + i|0\rangle_a |1\rangle_b\right] = \frac{1}{\sqrt{2}}\left[|1\rangle_a |0\rangle_b + i|0\rangle_a |1\rangle_b\right] \qquad N_{D1} \cdot \Psi = \begin{pmatrix} 0 \\ 0.707i \\ 0.707 \\ 0 \end{pmatrix}$$

Operating on Ψ with the D_2 number operator yields 0.

$$\left[\frac{C_a - iC_b}{\sqrt{2}} \cdot \frac{A_a + iA_b}{\sqrt{2}}\right] \frac{1}{\sqrt{2}}\left[|1\rangle_a |0\rangle_b + i|0\rangle_a |1\rangle_b\right] = 0 \qquad N_{D2} \cdot \Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$