The Kochen-Specker Theorem Illustrated Using A Three-Qubit GHZ Spin System

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In the '90s N. David Mermin published two articles in the general physics literature (*Physics Today*, June 1990; *American Journal of Physics*, August 1990) on the Greenberger-Horne-Zeilinger (GHZ) gedanken experiment (*American Journal of Physics*, December 1990; *Nature*, 3 February 2000) involving three spin-1/2 particles that illustrated the clash between local realism and the quantum view of reality for the quantum nonspecialist. The purpose of this tutorial is to use the GHZ example to illustrate the Kochen-Specker (KS) theorem by stripping away the use of the three-spin wave function in the analysis of the thought experiment.

The KS theorem asserts that no noncontextual (NC) hidden variable (HV) model (NCHV) can agree with the measurement predictions of quantum theory for Hilbert space dimensions greater than 2. The problem dealt with here is, of course, three-dimensional.

The three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. It will be shown that a consideration of spin measurements (in units of h/4π) in the x- and y-directions reveals the impossibility of assigning values to the spin observables independent of measurement.

The x- and y-direction spin operators are the Pauli matrices: $\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The eigenvalues of the Pauli matrices are +/- 1: $\text{eigenvals}(\sigma_x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\text{eigenvals}(\sigma_y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The following operators represent the measurement protocols for spins 1, 2 and 3.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \qquad \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

The tensor matrix product, also known as the Kronecker product, is available in Mathcad. The four operators in tensor format are formed as follows.

$$\sigma_{xyy} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_y, \sigma_y)) \qquad \sigma_{yxy} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_x, \sigma_y))$$

$$\sigma_{yyx} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_y, \sigma_x)) \qquad \sigma_{xxx} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_x, \sigma_x))$$

These operators mutually commute, meaning that they can be assigned simultaneous eigenstates with simultaneous eigenvalues.

$$\sigma_{xyy} \cdot \sigma_{yxy} - \sigma_{yxy} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{xyy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{xyy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{xyy} \rightarrow 0$$

$$\sigma_{yxy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{yxy} \rightarrow 0 \qquad \sigma_{yxy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yxy} \rightarrow 0 \qquad \sigma_{yyx} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yyx} \rightarrow 0$$

The next step is to compare the matrix for the product of the first three operators ($\sigma_{xyy} \cdot \sigma_{yxy} \cdot \sigma_{yyx}$) with that of the fourth (σ_{xxx}).

$$\sigma_{xyy}\cdot\sigma_{yxy}\cdot\sigma_{yyx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \sigma_{xxx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This indicates the following relationship between the four operators and leads quickly to a refutation of the concept of noncontextual, hidden values for quantum mechanical observables.

$$\left(\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3\right)\left(\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3\right)\left(\sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3\right) = -\left(\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3\right)$$

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes that the x- and y-components of the spin have definite values prior to measurement. This position leads to a contradiction with the above result. There is no way to assign eigenvalues (+/-1) to the operators that is consistent with the above result.

Concentrating on the operator on the left side, we notice that there is a σ_y measurement on the first spin in the second and third operator. If the spin state is well-defined before measurement and independent of what's being measured on spins 2 and 3 (the context, either $\sigma_x\sigma_y$ or $\sigma_y\sigma_x$) those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1. There is a σ_y measurement on the second spin in terms one and three. By similar arguments those results will lead to a product of +1 also. Finally there is a σ_y measurement on the third spin in terms one and two. By similar arguments those results will also lead to a product of +1. Incorporating these observations into the expression above leads to the following contradiction.

$$\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3 = -\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

Although it wasn't required here's the spin state vector and calculations using it that are consistent with previous analysis.

$$\Psi := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}^T$$

$$\Psi^T\cdot\sigma_{xyy}\cdot\sigma_{yxy}\cdot\sigma_{yyx}\cdot\Psi = 1 \qquad \Psi^T\cdot\sigma_{xxx}\cdot\Psi = -1$$

A Quirk Quantum Simulator (algassert.com/quirk) circuit for calculating these expectation values is shown below.

$\vert 1\rangle$	H	•		X		Y		Y		X		•	H		$\vert 1\rangle$
		|										|			
$\vert 0\rangle$		⊕	•	Y	or	X	or	Y	or	X	•	⊕		±	$\vert 0\rangle$
			|								|				
$\vert 0\rangle$			⊕	Y		Y		X		X	⊕				$\vert 0\rangle$