Gram-Schmidt Orthogonalization

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In this exercise the Gram-Schmidt method will be used to create an orthonormal basis set from the following vectors which are neither normalized nor orthogonal.

$$u1 := \begin{pmatrix} 1+i \\ 1 \\ i \end{pmatrix} \qquad u2 := \begin{pmatrix} i \\ 3 \\ 1 \end{pmatrix} \qquad u3 := \begin{pmatrix} 0 \\ 28 \\ 0 \end{pmatrix}$$

Demonstrate that the vectors are not normalized and are not orthogonal.

$$\left(\overline{u1}\right)^T \cdot u1 = 4 \qquad \left(\overline{u2}\right)^T \cdot u2 = 11 \qquad \left(\overline{u3}\right)^T \cdot u3 = 784$$

$$\left(\overline{u1}\right)^T \cdot u2 = 4 \qquad \left(\overline{u1}\right)^T \cdot u3 = 28 \qquad \left(\overline{u2}\right)^T \cdot u3 = 84$$

Using the first vector make u2 orthogonal to it by subtracting its projection on u1.

$$u2 := u2 - \frac{\left(\overline{u1}\right)^T \cdot u2}{\left(\overline{u1}\right)^T \cdot u1} \cdot u1$$

Make u3 orthogonal to u1 and u2 by subtracting its projection on u1 and u2.

$$u3 := u3 - \frac{\left(\overline{u1}\right)^T \cdot u3}{\left(\overline{u1}\right)^T \cdot u1} \cdot u1 - \frac{\left(\overline{u2}\right)^T \cdot u3}{\left(\overline{u2}\right)^T \cdot u2} \cdot u2$$

Finally, normalize the new orthogonal vectors.

$$u1 := \frac{u1}{\sqrt{\left(\overline{u1}\right)^T \cdot u1}} \qquad u2 := \frac{u2}{\sqrt{\left(\overline{u2}\right)^T \cdot u2}} \qquad u3 := \frac{u3}{\sqrt{\left(\overline{u3}\right)^T \cdot u3}}$$

Demonstrate that an orthonormal basis set has been created.

$$\left(\overline{u1}\right)^T \cdot u1 = 1 \qquad \left(\overline{u2}\right)^T \cdot u2 = 1 \qquad \left(\overline{u3}\right)^T \cdot u3 = 1$$

$$\left(\overline{u1}\right)^T \cdot u2 = 0 \qquad \left(\overline{u1}\right)^T \cdot u3 = 0 \qquad \left(\overline{u2}\right)^T \cdot u3 = 0$$

Display the orthonormal basis set.

$$u1 = \begin{pmatrix} 0.5+0.5i \\ 0.5 \\ 0.5i \end{pmatrix} \qquad u2 = \begin{pmatrix} -0.378 \\ 0.756 \\ 0.378-0.378i \end{pmatrix} \qquad u3 = \begin{pmatrix} 0.085-0.592i \\ 0.423 \\ -0.676+0.085i \end{pmatrix}$$