Greenberger-Horne-Zeilinger (GHZ) Entanglement and Local Realism Analyzed Using Tensor Algebra

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This tutorial analyses experimental results on a GHZ entanglement reported by Anton Zeilinger and collaborators in the 3 February 2000 issue of *Nature* (pp. 515-519) using tensor algebra. The GHZ experiment employs three-photon entanglement to provide a stunning attack on local realism.

First some definitions:

Realism - experiments yield values for properties that exist independent of experimental observation

Locality - the experimental results obtained at location *A* at time *t*, do not depend on the results at some other location *B* at time *t*.

H/V = horizontal/vertical linear polarization. **R/L** = right/left circular polarization. **H'/V'** rotated by 45° with respect to **H/V**.

Next the vector representations of the various photon polarization states:

$$H := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad H' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad V' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad L := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad R := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \quad N := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Some additional matrices needed to form state vectors in Mathcad:

$HN := \text{augment}(H, N)$ $\quad VN := \text{augment}(V, N)$ $\quad RN := \text{augment}(R, N)$

$LN := \text{augment}(L, N)$ $\quad H'N := \text{augment}(H', N)$ $\quad V'N := \text{augment}(V', N)$

The operators associated with linear and circular polarization:

$$H'V' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad RL := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The initial GHZ three-photon entangled state: $\Psi = \frac{1}{\sqrt{2}} \cdot (H_1 \cdot H_2 \cdot H_3 + V_1 \cdot V_2 \cdot V_3)$

Initial state expressed in tensor format:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Constructing the initial state using Mathcad:

$$\Psi := \frac{1}{\sqrt{2}} \cdot \mathrm{submatrix}(\mathrm{kronecker}(HN, \mathrm{kronecker}(HN, HN)) + \mathrm{kronecker}(VN, \mathrm{kronecker}(VN, VN)), 1, 8, 1, 1)$$

$$\Psi^{T} = (\,0.707 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.707\,)$$

After preparation of the initial GHZ state (see figure 1 in the reference cited above), polarization measurements are performed on the three photons. Zeilinger and collaborators use **y** to stand for a circular polarization measurement and **x** for a linear polarization measurement. Initially they perform circular polarization measurements on two of the photons and a linear polarization measurement on the other photon. The quantum mechanically predicted results and actual experimental measurements are given below.

yyx - experiment

The **yyx** operator is calculated and it is shown that Ψ is an eigenstate of the operator with eigenvalue -1.

$$yyx := \mathrm{kronecker}(RL, \mathrm{kronecker}(RL, H'V')) \qquad \Psi^{T} \cdot yyx \cdot \Psi = -1$$

Each measurment (R/L or H'/V') has two possible outcomes so, in principle, there could be 8 possible results. However, quantum mechanics predicts that only four equally probable outcomes are possible. This is because the eigenvalues of the individual results must be consistent with the eigenvalue the total operator. As shown below, the eigenvalues for H' and R are +1, and for V' and L they are -1.

$$H'V' \cdot H' = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad H'V' \cdot V' = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix} \quad RL \cdot R = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} \quad RL \cdot L = \begin{pmatrix} -0.707 \\ 0.707i \end{pmatrix}$$

Thus quantum mechanics predicts that RRV' (++-), LRH' (-++), RLH' (+-+), and LLV' (---) should be observed, but RRH' (+++), LRV' (-+-), RLV' (+--) and LLH' (--+) should not be observed. This prediction is in agreement with the experimental results shown in Figure 1 to within experimental error.

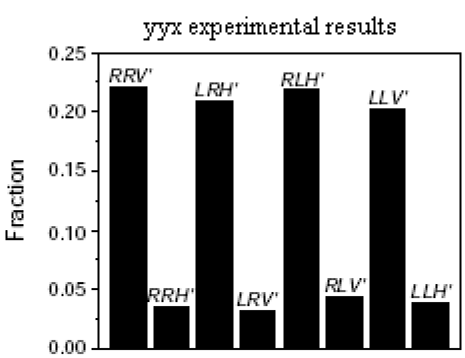

Figure 1

Confirm the first two results in Figure 1.

$$RRV' := \mathrm{submatrix}(\mathrm{kronecker}(RN, \mathrm{kronecker}(RN, V'N)), 1, 8, 1, 1)$$

$$(|RRV' \cdot \Psi|)^2 = 0.25$$

$$RRH' := \mathrm{submatrix}(\mathrm{kronecker}(RN, \mathrm{kronecker}(RN, H'N)), 1, 8, 1, 1)$$

$$(|RRH' \cdot \Psi|)^2 = 0$$

For the two remaining experiments in this class (**yxy** and **xyy**), the agreement between theoretical prediction and experimental results is basically the same.

While these agreements between quantum mechanics and experiment are impressive they do not directly challenge the local realist position. As will be shown later that will be accomplished by a fourth experiment involving the measurement of the linear polarization on all three photons - the **xxx** experiment.

yxy - experiment

The **yxy** operator is calculated and it is shown that Ψ is an eigenstate of the operator with eigenvalue -1.

$$yxy := \text{kronecker}(RL, \text{kronecker}(H'V', RL)) \qquad \Psi^T \cdot yxy \cdot \Psi = -1$$

Using reasoning identical to the **yyx** experiment, quantum mechanics predicts that RV'R (+-+), LH'R (-++), RH'L (++-), and LV'L (---) should be observed, but RH'R (+++), LV'R (--+), RV'L (+--) and LH'L (-+-) should not be observed. This prediction is in agreement with the experimental results shown in Figure 2 to within experimental error.

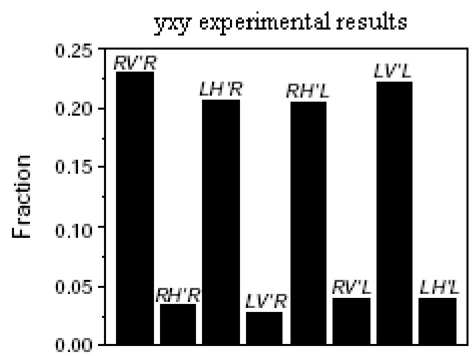

Figure 2

Confirm the first two results in Figure 2.

$$RV'R := \text{submatrix}(\text{kronecker}(RN, \text{kronecker}(V'N, RN)), 1, 8, 1, 1)$$

$$\left(\left|RV'R \cdot \Psi\right|\right)^2 = 0.25$$

$$RH'R := \text{submatrix}(\text{kronecker}(RN, \text{kronecker}(H'N, RN)), 1, 8, 1, 1)$$

$$\left(\left|RH'R \cdot \Psi\right|\right)^2 = 0$$

xyy - experiment

The **xyy** operator is calculated and it is shown that Ψ is an eigenstate of the operator with eigenvalue -1.

$$xyy := \text{kronecker}(H'V', \text{kronecker}(RL, RL)) \qquad \Psi^T \cdot xyy \cdot \Psi = -1$$

Using reasoning identical to the previous experiments, quantum mechanics predicts that V'RR (-++), H'LR (+-+), H'RL (++-), and V'LL (---) should be observed, but H'RR (+++), V'LR (--+), V'RL (-+-) and H'LL (+--) should not be observed. This prediction is in agreement with the experimental results shown in Figure 3 to within experimental error.

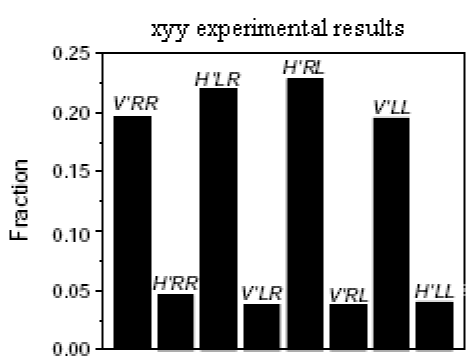

Figure 3

Confirm the first two results in Figure 3.

$$V'RR := \text{submatrix}(\text{kronecker}(V'N, \text{kronecker}(RN, RN)), 1, 8, 1, 1)$$

$$\left(\left|V'RR \cdot \Psi\right|\right)^2 = 0.25$$

$$H'RR := \text{submatrix}(\text{kronecker}(H'N, \text{kronecker}(RN, RN)), 1, 8, 1, 1)$$

$$\left(\left|H'RR \cdot \Psi\right|\right)^2 = 0$$

Now for the critical experiment.

xxx - experiment

The **xxx** operator is calculated and it is shown that Ψ is an eigenstate of the operator with eigenvalue **+1**.

$$xxx := \text{kronecker}(H'V', \text{kronecker}(H'V', H'V')) \qquad \Psi^T \cdot xxx \cdot \Psi = 1$$

Using reasoning identical to the previous experiments, quantum mechanics predicts that only H'V'V' (+--), V'H'V' (-+-), V'V'H' (--+), and H'H'H' (+++) should be observed. This prediction is displayed graphically in Figure 4.

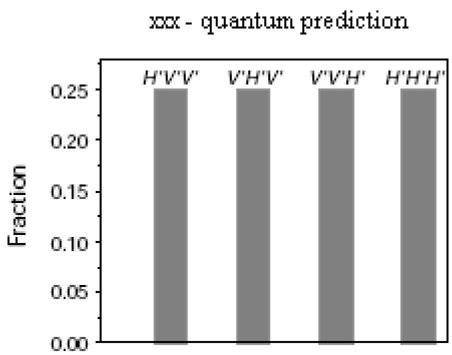

Figure 4

QM calculation for xxx experiment.

$$V'V'V' := \text{submatrix}(\text{kronecker}(V'N, \text{kronecker}(V'N, V'N)), 1, 8, 1, 1)$$

$$(|V'V'V' \cdot \Psi|)^2 = 0$$

$$H'V'V' := \text{submatrix}(\text{kronecker}(H'N, \text{kronecker}(V'N, V'N)), 1, 8, 1, 1)$$

$$(|H'V'V' \cdot \Psi|)^2 = 0.25$$

According to local realism the experimental outcome should be as shown in Figure 5. To explain how the local realist prediction is derived, we recall that this position assumes that physical properties exist independent of measurement. Previously it has been shown that eigenvalues of the three-photon operators according to quantum mechanics are,

$$y_1 \cdot y_2 \cdot x_3 = y_1 \cdot x_2 \cdot y_3 = x_1 \cdot y_2 \cdot y_3 = -1 \qquad x_1 \cdot x_2 \cdot x_3 = 1$$

It follows that,

$$(y_1 \cdot y_2 \cdot x_3) \cdot (y_1 \cdot x_2 \cdot y_3) \cdot (x_1 \cdot y_2 \cdot y_3) = -1$$

However, this means that $x_1x_2x_3 = -1$, because $y_1y_1 = y_2y_2 = y_3y_3 = 1$. (All of the three-photon operators commute, and therefore can have simultaneous eigenvalues. The commutation relations are shown in the Appendix.) According to the local realist view there are four ways to achieve this result ($x_1x_2x_3 = -1$) V'V'V', H'H'V', H'V'H' and V'H'H'.

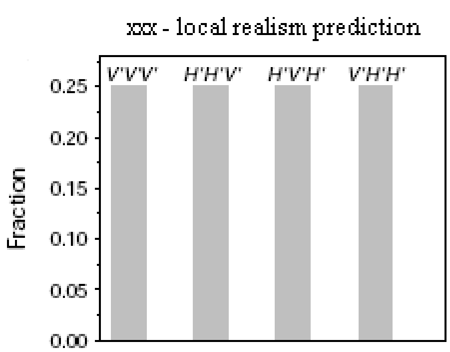

Figure 5

LR results are contradicted by QM calculation.

$$V'V'V' := \text{submatrix}(\text{kronecker}(V'N, \text{kronecker}(V'N, V'N)), 1, 8, 1, 1)$$

$$(|V'V'V' \cdot \Psi|)^2 = 0$$

$$H'H'V' := \text{submatrix}(\text{kronecker}(H'N, \text{kronecker}(H'N, V'N)), 1, 8, 1, 1)$$

$$(|H'H'V' \cdot \Psi|)^2 = 0$$

Figure 6 shows that none local realistic predictions are observed at a statistically meaningful level in tl GHZ experiment. The quantum mechanical prediction for the **xxx** experiment agrees with experiment, the local realist prediction doesn't.

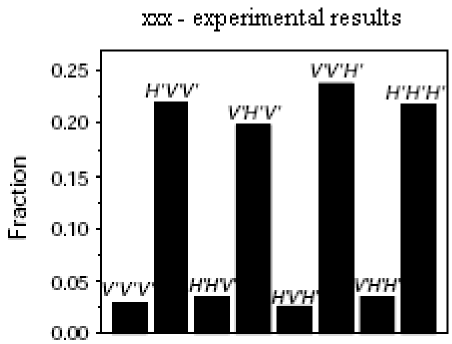

Figure 6

QM calculations confirmed by experimental results.

V'V'V' := submatrix(kronecker(V'N, kronecker(V'N, V'N)), 1, 8, 1, 1)

$$(|V'V'V' \cdot \Psi|)^2 = 0$$

H'V'V' := submatrix(kronecker(H'N, kronecker(V'N, V'N)), 1, 8, 1, 1)

$$(|H'V'V' \cdot \Psi|)^2 = 0.25$$

For the **xxx** experiment, the mathematical predictions of quantum mechanics (Figure 4) and the local realist view (Figure 5) when compared with the actual experimental results (Figure 6) present a convincing refutation of local realism.

Appendix

Demonstration that the three-photon operators commute, allowing for simultaneous eigenvalues.

$$xyy \cdot yxy - yxy \cdot xyy \rightarrow 0 \qquad xyy \cdot yyx - yyx \cdot xyy \rightarrow 0 \qquad xyy \cdot xxx - xxx \cdot xyy \rightarrow 0$$

$$yxy \cdot yyx - yyx \cdot yxy \rightarrow 0 \qquad yxy \cdot xxx - xxx \cdot yxy \rightarrow 0 \qquad yyx \cdot xxx - xxx \cdot yyx \rightarrow 0$$