GHZ Four-Photon Entanglement Analyzed Using Tensor Algebra

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This tutorial analyzes the results reported in "Experimental Violation of Local Realism by Four-Photon GHZ Entanglement" by Zhao, *et al*. and published in *Physical Review Letters* on October 31, 2003.

The null vector and required photon polarization states:

$$N := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad H := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad H' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad V' := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad L := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad R := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$$

To facilitate tensor vector multiplication the polarization states are stored in the left column of a 2x2 matrix using the null vector.

$$H := \text{augment}(H, N) \quad V := \text{augment}(V, N) \quad H' := \text{augment}(H', N) \quad V' := \text{augment}(V', N) \quad R := \text{augment}(R, N) \quad L := \text{augment}(L, N)$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H' = \begin{pmatrix} 0.707 & 0 \\ 0.707 & 0 \end{pmatrix} \quad V' = \begin{pmatrix} 0.707 & 0 \\ -0.707 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 0.707 & 0 \\ 0.707i & 0 \end{pmatrix} \quad L = \begin{pmatrix} 0.707 & 0 \\ -0.707i & 0 \end{pmatrix}$$

Operators: $H'V' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad RL := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Eigenvalues of various polarization states:

$$H'V' \cdot H' = \begin{pmatrix} 0.707 & 0 \\ 0.707 & 0 \end{pmatrix} \quad H'V' \cdot V' = \begin{pmatrix} -0.707 & 0 \\ 0.707 & 0 \end{pmatrix} \quad RL \cdot R = \begin{pmatrix} 0.707 & 0 \\ 0.707i & 0 \end{pmatrix} \quad RL \cdot L = \begin{pmatrix} -0.707 & 0 \\ 0.707i & 0 \end{pmatrix}$$

Eigenvalue = 1   Eigenvalue = −1   Eigenvalue = 1   Eigenvalue = −1

The initial GHZ four-photon entangled state: $\Psi = \frac{1}{\sqrt{2}} \cdot \left(H_1 \cdot V_2 \cdot V_3 \cdot H_4 + V_1 \cdot H_2 \cdot H_3 \cdot V_4\right)$

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

Initial state set up in tensor format.

$$\Psi_i := \frac{1}{\sqrt{2}} \cdot (\text{submatrix}(\text{kronecker}(H, \text{kronecker}(V, \text{kronecker}(V, H))) + \text{kronecker}(V, \text{kronecker}(H, \text{kronecker}(H, V))), 1, 16, 1, 1))$$

$$\Psi_i^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0 & 0 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The authors initially consider three measurements which are summarized below. It is shown that the inital photon state is an eigenstate of each of the operators with eigenvalue +1. In the operators x refers to a linear polarization measurement (H'V') and y refers to a circular polarization measurement (RL). The experimental results are reported in Figure 3 of the paper. Below quantum mechanical (QM) calculations (predictions) are compared with experimental outcomes. QM agrees with experiment.

The following calculations are facilitated by the following general expression for the measurement eigenstates.

$$\Psi(a,b,c,d) := \text{submatrix}(\text{kronecker}(a,\text{kronecker}(b,\text{kronecker}(c,d))),1,16,1,1)$$

σ_{xxxx} experiment:

Operator: $\sigma_{xxxx} := \text{kronecker}(H'V',\text{kronecker}(H'V',\text{kronecker}(H'V',H'V')))$ Eigenvalue/Expectation Value: $\Psi_i^T \cdot \sigma_{xxxx} \cdot \Psi_i = 1$

Observed: H'H'H'H', H'H'V'V', H'V'H'V', H'V'V'H', V'H'H'V', V'H'V'H', V'V'H'H' and V'V'V'V'.

$\left(\left|\Psi(H',H',H',H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',H',V',V')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',V',V',H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',V',V',H')^T \cdot \Psi_i\right|\right)^2 = 0.125$

$\left(\left|\Psi(V',H',H',V')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',H',V',H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',V',H',H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',V',V',V')^T \cdot \Psi_i\right|\right)^2 = 0.125$

σ_{xyxy} experiment:

Operator: $\sigma_{xyxy} := \text{kronecker}(H'V',\text{kronecker}(RL,\text{kronecker}(H'V',RL)))$ Eigenvalue/Expectation Value: $\Psi_i^T \cdot \sigma_{xyxy} \cdot \Psi_i = 1$

Observed: H'RH'R, H'RV'L, H'LH'L, H'LV'R, V'RH'L, V'RV'R, V'LH'R and V'LV'L.

$\left(\left|\Psi(H',R,H',R)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',R,V',L)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',L,H',L)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',L,V',R)^T \cdot \Psi_i\right|\right)^2 = 0.125$

$\left(\left|\Psi(V',R,H',L)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',R,V',R)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',L,H',R)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',L,V',L)^T \cdot \Psi_i\right|\right)^2 = 0.125$

σ_{xxyy} experiment:

Operator: $\sigma_{xxyy} := \text{kronecker}(H'V',\text{kronecker}(H'V',\text{kronecker}(RL,RL)))$ Eigenvalue/Expectation Value: $\Psi_i^T \cdot \sigma_{xxyy} \cdot \Psi_i = 1$

Observed: H'H'RR, H'H'LL,H'V'RL, H'V'LR, V'H'RL, V'H'LR, V'V'RR and V'V'LL.

$\left(\left|\Psi(H',H',R,R)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',H',L,L)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',V',R,L)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',V',L,R)^T \cdot \Psi_i\right|\right)^2 = 0.125$

$\left(\left|\Psi(V',H',R,L)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',H',L,R)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',V',R,R)^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',V',L,L)^T \cdot \Psi_i\right|\right)^2 = 0.125$

This analysis shows that quantum mechanics (QM) is in agreement with experimental results. The next step is to perform an experiment that shows that local realism (LR) is not in agreement with experimental results.

The fact that the eigenvalues of the individual operators examined above is +1, guarrantees that the same is true for their product.

$$(x_1x_2x_3x_4)(x_1y_2x_3y_4)(x_1x_2y_3y_4) = 1$$

Local realism assumes that physical properties exist independent of measurement. Because commuting operators have simultaneous eigenvalues $x_1x_1 = x_2x_2 = x_3x_3 = y_4y_4$. It follows that,

$$(x_1y_2y_3x_4) = 1$$

The following results are consistent with this local realism analysis: H'RRH', H'RLV', H'LRV', H'LLH', V'RRV', V'RLH', V'LRH' and V'LLV'. As shown below, this is in complete disagreement with quantum mechanics and the experimental data. QM shows that the eigenvalue of the operator is actually -1, and, furthermore none of LR predicted results are observed. QM, however, is in agreement with the experitmental results.

σ_{xyyx} experiment:

Operator: $\sigma_{xyyx} := \text{kronecker}(H'V', \text{kronecker}(RL, \text{kronecker}(RL, H'V')))$ Eigenvalue/Expectation Value: $\Psi_i^T \cdot \sigma_{xyyx} \cdot \Psi_i = -1$

Observed: H'RRV', H'RLH', H'LRH', H'LLV', V'RRH', V'RLV', V'LRV' and V'LLH'.

$\left(\left|\Psi(H',R,R,V')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',R,L,H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',L,R,H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(H',L,L,V')^T \cdot \Psi_i\right|\right)^2 = 0.125$

$\left(\left|\Psi(V',R,R,H')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',R,L,V')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',L,R,V')^T \cdot \Psi_i\right|\right)^2 = 0.125$ $\left(\left|\Psi(V',L,L,H')^T \cdot \Psi_i\right|\right)^2 = 0.125$

Appendix

All four operators commute with each other allowing them to have simultaneous eigenvalues.

$\sigma_{xxxx} \cdot \sigma_{xyxy} - \sigma_{xyxy} \cdot \sigma_{xxxx} \rightarrow 0$ $\sigma_{xxxx} \cdot \sigma_{xyyx} - \sigma_{xyyx} \cdot \sigma_{xxxx} \rightarrow 0$ $\sigma_{xxxx} \cdot \sigma_{xxyy} - \sigma_{xxyy} \cdot \sigma_{xxxx} \rightarrow 0$

$\sigma_{xxyy} \cdot \sigma_{xyxy} - \sigma_{xyxy} \cdot \sigma_{xxyy} \rightarrow 0$ $\sigma_{xxyy} \cdot \sigma_{xyyx} - \sigma_{xyyx} \cdot \sigma_{xxyy} \rightarrow 0$ $\sigma_{xyxy} \cdot \sigma_{xyyx} - \sigma_{xyyx} \cdot \sigma_{xyxy} \rightarrow 0$