Simulation of a GHZ Gedanken Experiment

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This tutorial analyzes a GHZ (Greenberger-Horne-Zeilinger) thought experiment involving three spin-1/2 particles that illustrates the clash between local realism and the quantum view of reality. The analysis consists of three parts: a traditional theoretical analysis, a quantum computer simulation, and an analysis based local realism. In the 1990s N. David Mermin published two articles in the general physics literature (*Physics Today*, June 1990; *American Journal of Physics*, August 1990) on the GHZ gedanken experiment. I drew heavily on these articles in developing this tutorial.

Theoretical Analysis

We begin with a quantum mechanical analysis of a version of the GHZ thought experiment suggested by Mermin. The initial state, Ψ, and Mermin's operator are as follows.

$$\Psi := \frac{1}{\sqrt{2}} \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ i)^T$$

$$\widehat{M} = \sigma_y \sigma_x \sigma_x + \sigma_x \sigma_y \sigma_x + \sigma_x \sigma_x \sigma_y - \sigma_y \sigma_y \sigma_y$$

The Pauli σ_x and σ_y individual spin operators and their composites required for Mermin's operator:

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_{xxy} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_x, \sigma_y)) \qquad \sigma_{xyx} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_y, \sigma_x))$$

$$\sigma_{yxx} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_x, \sigma_x)) \qquad \sigma_{yyy} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_y, \sigma_y))$$

The composite operators commute (σ_x and σ_y don't) and therefore can have simultaneous eigenvalues.

$$\sigma_{xxy} \cdot \sigma_{xyx} - \sigma_{xyx} \cdot \sigma_{xxy} \rightarrow 0 \qquad \sigma_{xxy} \cdot \sigma_{yxx} - \sigma_{yxx} \cdot \sigma_{xxy} \rightarrow 0 \qquad \sigma_{xxy} \cdot \sigma_{yyy} - \sigma_{yyy} \cdot \sigma_{xxy} \rightarrow 0$$

$$\sigma_{xyx} \cdot \sigma_{yxx} - \sigma_{yxx} \cdot \sigma_{xyx} \rightarrow 0 \qquad \sigma_{xyx} \cdot \sigma_{yyy} - \sigma_{yyy} \cdot \sigma_{xyx} \rightarrow 0 \qquad \sigma_{yxx} \cdot \sigma_{yyy} - \sigma_{yyy} \cdot \sigma_{yxx} \rightarrow 0$$

The calculations of various expectation values based on the proposed initial state and operator:

$$M := \sigma_{yxx} + \sigma_{xyx} + \sigma_{xxy} - \sigma_{yyy}$$

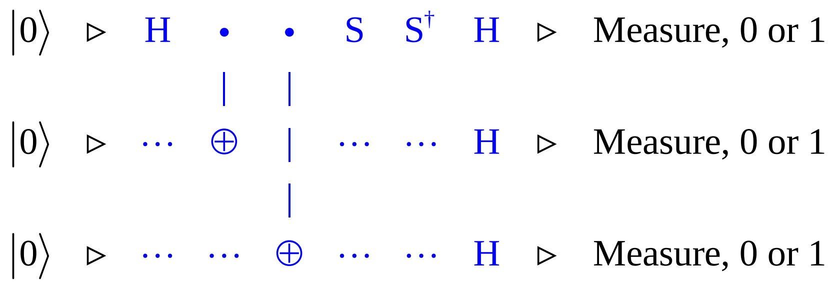
$$(\overline{\Psi})^T \cdot \sigma_{yxx} \cdot \Psi = 1 \qquad (\overline{\Psi})^T \cdot \sigma_{xyx} \cdot \Psi = 1 \qquad (\overline{\Psi})^T \cdot \sigma_{xxy} \cdot \Psi = 1 \qquad (\overline{\Psi})^T \cdot \sigma_{yyy} \cdot \Psi = -1 \qquad (\overline{\Psi})^T \cdot M \cdot \Psi = 4$$

The key result is that the expectation value for **M** is 4. Subsequently it will be shown that a quantum simulator circuit is in agreement with this result but that local realism is not.

Quantum Computer Simulation

"Quantum simulation is a process in which a quantum computer simulates another quantum system. Because of the various types of quantum weirdness, classical computers can simulate quantum systems only in a clunky, inefficient way. But because a quantum computer is itself a quantum system, capable of exhibiting the full repertoire of quantum weirdness, it can efficiently simulate other quantum systems. **The resulting simulation can be so accurate that the behavior the computer will be indistinguishable from the behavior of the simulated system itself.**" (Seth Lloyd, Programming the Universe, page 149.)

This is the most important part of the tutorial - a demonstration that the thought experiment can be simulated using the quantum circuit shown below which can be found at: arXiv:1712.05642v2; "Five Experimental Tests on the 5-qubit IBM Quantum Computer," Diego Garcia-Martin and German Sierra.



The required matrix operators and the build-up of the quantum circuit:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad S' := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad CnNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

HII := kronecker(H, kronecker(I, I))   CNOTI := kronecker(CNOT, I)   SII := kronecker(S, kronecker(I, I))

S'S'S' := kronecker(S', kronecker(S', S'))   HHH := kronecker(H, kronecker(H, H))   S'II := kronecker(S', kronecker(I, I))

IS'I := kronecker(I, kronecker(S', I))   IIS' := kronecker(I, kronecker(I, S'))

First it is demonstrated that the first four steps of the circuit create the initial state.

$$\left[SII \cdot CnNOT \cdot CNOTI \cdot HII \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T\right]^T = (0.707\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.707i)$$

The complete circuit shows the simulation for the expectation value of $\sigma_y\sigma_x\sigma_x$. The presence of S' on a line before the final H gates indicates the measurement of the σ_y. The subsequent simulations show the presence of S' on the middle and last line, and finally on all three lines for the simulation of the expectation value for $\sigma_y\sigma_y\sigma_y$.

Eigenvalue of |0> = +1; eigenvalue of |1> = -1

$$\langle \sigma_y\sigma_x\sigma_x \rangle = 1 \qquad HHH \cdot S'II \cdot SII \cdot CnNOT \cdot CNOTI \cdot HII \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$ = 1/2[|000> + |011> + |101> + |110>]

Given the eigenvalue assignments above the expectation value associated with this outcome is 1/4[(1)(1)(1)+(1)(-1)(-1)+(-1)(1)(-1)+(-1)(-1)(1)] = 1. Note that 1/2 is the probability amplitude for the product state. Therefore the probability of each member of the superposition being observed is 1/4. Similar reasoning is used for the remaining simulations.

$$\left\langle \sigma_x \sigma_y \sigma_x \right\rangle = 1 \qquad \mathrm{HHH \cdot IS'I \cdot SII \cdot CnNOT \cdot CNOTI \cdot HII} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

= 1/2[|000> + |011> + |101> + |110>]

$$\left\langle \sigma_x \sigma_x \sigma_y \right\rangle = 1 \qquad \mathrm{HHH \cdot IIS' \cdot SII \cdot CnNOT \cdot CNOTI \cdot HII} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

= 1/2[|000> + |011> + |101> + |110>]

$$\left\langle \sigma_y \sigma_y \sigma_y \right\rangle = -1 \qquad \mathrm{HHH \cdot S'S'S' \cdot SII \cdot CnNOT \cdot CNOTI \cdot HII} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \end{pmatrix}$$

= 1/2[|001> + |010> + |100> + |111>]

$$\left\langle M \right\rangle = \left\langle \sigma_y \sigma_x \sigma_x \right\rangle + \left\langle \sigma_x \sigma_y \sigma_x \right\rangle + \left\langle \sigma_x \sigma_x \sigma_y \right\rangle - \left\langle \sigma_y \sigma_y \sigma_y \right\rangle = 4$$

The simulation is in exact agreement with the initial theoretical analysis.

EPR Local Realistic Analysis

Local realism asserts that objects have definite properties independent of measurement. In this experiment it assumes that the x- and y-components of spin have definite values prior to measurement. This position leads to a contradiction with the quantum mechanical calculation and the simulation. There is no way to assign consistent eigenvalues (+/-1) to the results of the individual spin measurements that is consistent with the quantum mechanical result. Using a variety of possible x- and y-spin values shows that local realism predicts that $M \le 2$, in sharp disagreement with the quantum mechanical result of $M = 4$.

$Sx1 := 1 \qquad Sx2 := 1 \qquad Sx3 := 1 \qquad Sy1 := 1 \qquad Sy2 := 1 \qquad Sy3 := 1$

$$M := Sy1 \cdot Sx2 \cdot Sx3 + Sx1 \cdot Sy2 \cdot Sx3 + Sx1 \cdot Sx2 \cdot Sy3 - Sy1 \cdot Sy2 \cdot Sy3 = 2$$

$Sx1 := -1 \quad Sx2 := 1 \quad Sx3 := 1 \quad Sy1 := 1 \quad Sy2 := -1 \quad Sy3 := -1$

$M := Sy1 \cdot Sx2 \cdot Sx3 + Sx1 \cdot Sy2 \cdot Sx3 + Sx1 \cdot Sx2 \cdot Sy3 - Sy1 \cdot Sy2 \cdot Sy3 = 2$

$Sx1 := 1 \quad Sx2 := -1 \quad Sx3 := 1 \quad Sy1 := -1 \quad Sy2 := 1 \quad Sy3 := -1$

$M := Sy1 \cdot Sx2 \cdot Sx3 + Sx1 \cdot Sy2 \cdot Sx3 + Sx1 \cdot Sx2 \cdot Sy3 - Sy1 \cdot Sy2 \cdot Sy3 = 2$

$Sx1 := -1 \quad Sx2 := 1 \quad Sx3 := -1 \quad Sy1 := -1 \quad Sy2 := 1 \quad Sy3 := -1$

$M := Sy1 \cdot Sx2 \cdot Sx3 + Sx1 \cdot Sy2 \cdot Sx3 + Sx1 \cdot Sx2 \cdot Sy3 - Sy1 \cdot Sy2 \cdot Sy3 = 2$

$Sx1 := -1 \quad Sx2 := -1 \quad Sx3 := -1 \quad Sy1 := -1 \quad Sy2 := 1 \quad Sy3 := -1$

$M := Sy1 \cdot Sx2 \cdot Sx3 + Sx1 \cdot Sy2 \cdot Sx3 + Sx1 \cdot Sx2 \cdot Sy3 - Sy1 \cdot Sy2 \cdot Sy3 = -2$

$Sx1 := 1 \quad Sx2 := 1 \quad Sx3 := 1 \quad Sy1 := -1 \quad Sy2 := -1 \quad Sy3 := -1$

$M := Sy1 \cdot Sx2 \cdot Sx3 + Sx1 \cdot Sy2 \cdot Sx3 + Sx1 \cdot Sx2 \cdot Sy3 - Sy1 \cdot Sy2 \cdot Sy3 = -2$

Alsina and Latorre ("Experimental test of Mermin inequalities on a five qubit quantum computer" available at arXiv:1605.04220V2) used the following alternative quantum circuit for the GHZ simulation. Here it is shown that it provides the same results for the $\sigma_y \sigma_y \sigma_y$ simulation.

```
|0⟩ ▷ ··· ··· H  •  H ··· S† H ▷ Measure, 0 or 1
                 |
|0⟩ ▷ H  •  H   |  ··· ··· S† H ▷ Measure, 0 or 1
          |      |
|0⟩ ▷ ··· ⊕ ···  ⊕  H  S   S† H ▷ Measure, 0 or 1
```

$IIS := kronecker(I, kronecker(I, S)) \quad HIH := kronecker(H, kronecker(I, H)) \quad HHI := kronecker(H, kronecker(H, I))$

$IHI := kronecker(I, kronecker(H, I)) \quad ICNOT := kronecker(I, CNOT)$

$$HHH \cdot S'S'S' \cdot IIS \cdot HIH \cdot CnNOT \cdot HHI \cdot ICNOT \cdot IHI \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \end{pmatrix}$$

= 1/2[|001> + |010> + |100> + |111>]

$$\langle \sigma_y \sigma_y \sigma_y \rangle = -1$$

The first six steps of this circuit generate the initial state.

$$\left[IIS \cdot HIH \cdot CnNOT \cdot HHI \cdot ICNOT \cdot IHI \cdot (1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0)^T \right]^T = (0.707 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0.707i)$$