Quantum Circuit for the Generation of GHZ States

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The following circuit generates the eight GHZ maximally entangled states. A similar NMR circuit can be found on page 287 of *The Quest for the Quantum Computer* by Julian Brown.

Index						
$\vert a\rangle$	$\triangleright$	H	$\bullet$	$\cdots$	$\triangleright$	G
			$\vert$			
$\vert b\rangle$	$\triangleright$	$\cdots$	$\oplus$	$\bullet$	$\triangleright$	H
				$\vert$		
$\vert c\rangle$	$\triangleright$	$\cdots$	$\cdots$	$\oplus$	$\triangleright$	Z

Given the quantum gates in matrix form the quantum circuit is formed using Kronecker multiplication.

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad ICNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$GHZ := \mathrm{kronecker}(I, CNOT) \cdot \mathrm{kronecker}(CNOT, I) \cdot \mathrm{kronecker}(H, \mathrm{kronecker}(I, I))$$

Using the index as input the quantum circuit generates the corresponding GHZ state.

$$G_0 := GHZ \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$G_1 := GHZ \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$G_2 := GHZ\cdot\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0.707\\0.707\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$G_3 := GHZ\cdot\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0.707\\0\\0\\0.707\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$G_4 := GHZ\cdot\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0.707\\0\\0\\0\\0\\0\\0\\-0.707\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$G_5 := GHZ\cdot\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0.707\\0\\0\\0\\0\\-0.707\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$G_6 := GHZ\cdot\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0.707\\-0.707\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$G_7 := GHZ \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.707 \\ 0 \\ 0 \\ -0.707 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

Given the following truth tables the operation of the circuit is followed algebraically for G_0 and G_2.

Identity $\begin{pmatrix} 0 & \text{to} & 0 \\ 1 & \text{to} & 1 \end{pmatrix}$ Hadamard $H = \begin{bmatrix} 0 & \text{to} & \frac{(0+1)}{\sqrt{2}} & \text{to} & 0 \\ 1 & \text{to} & \frac{(0-1)}{\sqrt{2}} & \text{to} & 1 \end{bmatrix}$ CNOT

Decimal	Binary	to	Binary	Decimal
0	00	to	00	0
1	01	to	01	1
2	10	to	11	3
3	11	to	10	2

$$|000\rangle$$

H ⊗ I ⊗ I

$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle|0\rangle = \frac{1}{\sqrt{2}}\left[|000\rangle + |100\rangle\right]$$

CNOT ⊗ I

$$\frac{1}{\sqrt{2}}\left[|000\rangle + |110\rangle\right]$$

I ⊗ CNOT

$$\frac{1}{\sqrt{2}}\left[|000\rangle + |111\rangle\right]$$

$$|010\rangle$$

H ⊗ I ⊗ I

$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|1\rangle|0\rangle = \frac{1}{\sqrt{2}}\left[|010\rangle + |110\rangle\right]$$

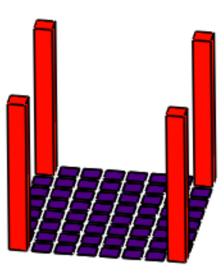
CNOT ⊗ I

$$\frac{1}{\sqrt{2}}\left[|010\rangle + |100\rangle\right]$$

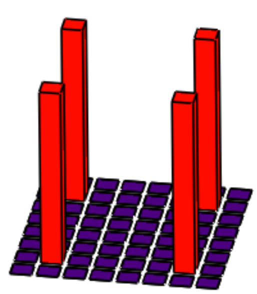
I ⊗ CNOT

$$\frac{1}{\sqrt{2}}\left[|011\rangle + |100\rangle\right]$$

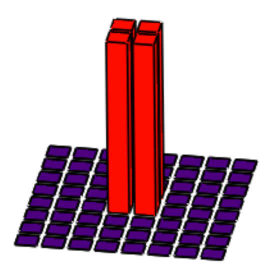
Next the GHZ density matrices are presented graphically.



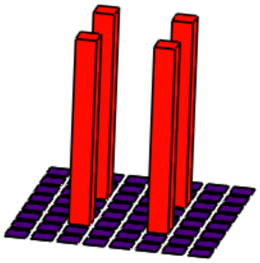
$G_0 \cdot G_0^T$



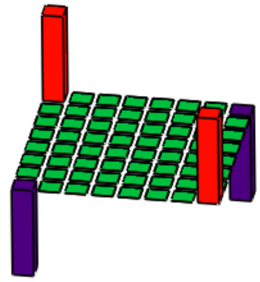
$G_1 \cdot G_1^T$



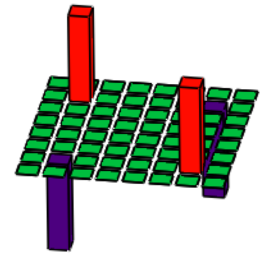
$G_2 \cdot G_2^T$



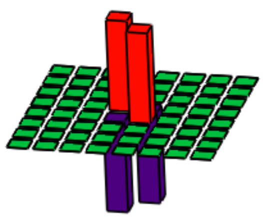
$G_3 \cdot G_3^T$



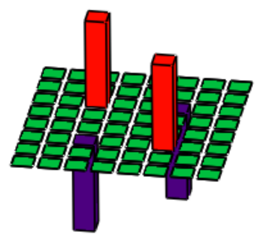
$G_4 \cdot G_4^T$



$G_5 \cdot G_5^T$



$G_6 \cdot G_6^T$



$G_7 \cdot G_7^T$

Using the GHZ states as input and running the circuit in reverse yields the indices.

$IGHZ := GHZ^{-1}$

$$IGHZ \cdot G_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad IGHZ \cdot G_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad IGHZ \cdot G_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad IGHZ \cdot G_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$IGHZ \cdot G_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad IGHZ \cdot G_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad IGHZ \cdot G_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad IGHZ \cdot G_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Possible direction for future development of this tutorial.

$Sx := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Sy := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $Sz := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$SzSzSz := \mathrm{kronecker}(Sz, \mathrm{kronecker}(Sz, Sz))$ $SxSxSx := \mathrm{kronecker}(Sx, \mathrm{kronecker}(Sx, Sx))$

$SySySy := \mathrm{kronecker}(Sy, \mathrm{kronecker}(Sy, Sy))$

$G_0{}^T \cdot SzSzSz \cdot G_0 = 0$ $G_0{}^T \cdot SxSxSx \cdot G_0 = 1$ $G_0{}^T \cdot SySySy \cdot G_0 = 0$

$G_1{}^T \cdot SzSzSz \cdot G_1 = 0$ $G_1{}^T \cdot SxSxSx \cdot G_1 = 1$ $G_1{}^T \cdot SySySy \cdot G_1 = 0$

$G_2{}^T \cdot SzSzSz \cdot G_2 = 0$ $G_2{}^T \cdot SxSxSx \cdot G_2 = 1$ $G_2{}^T \cdot SySySy \cdot G_2 = 0$

$G_3{}^T \cdot SzSzSz \cdot G_3 = 0$ $G_3{}^T \cdot SxSxSx \cdot G_3 = 1$ $G_3{}^T \cdot SySySy \cdot G_3 = 0$

$G_4{}^T \cdot SzSzSz \cdot G_4 = 0$ $G_4{}^T \cdot SxSxSx \cdot G_4 = -1$ $G_4{}^T \cdot SySySy \cdot G_4 = 0$

$G_5{}^T \cdot SzSzSz \cdot G_5 = 0$ $G_5{}^T \cdot SxSxSx \cdot G_5 = -1$ $G_5{}^T \cdot SySySy \cdot G_5 = 0$

$G_6{}^T \cdot SzSzSz \cdot G_6 = 0$ $G_6{}^T \cdot SxSxSx \cdot G_6 = -1$ $G_6{}^T \cdot SySySy \cdot G_6 = 0$

$G_7{}^T \cdot SzSzSz \cdot G_7 = 0$ $G_7{}^T \cdot SxSxSx \cdot G_7 = -1$ $G_7{}^T \cdot SySySy \cdot G_7 = 0$