A Surgical Refutation of the Local Realism Heresy

Frank Rioux
Emeritus Professor of Chemistry
CSB | SJU

Three photons are created in a single event (*Nature*, February 3, 2000 pp. 515-519) and move apart in the horizontal y-z plane. The goal of this exercise is to demonstrate that an analysis of measurements in the diagonal and circular polarization basis reveals the impossibility of assigning definite values to the polarization states of the photons prior to and independent of measurement.

First some definitions:

Realism - experiments yield values for properties that exist independent of experimental observation

Locality - the experimental results obtained at location *A* at time *t*, do not depend on the results at some other remote location *B* at time *t*.

The diagonal and circular polarization matrix operators: $D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $C := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The eigenvalues of the matrices are +/- 1: $\text{eigenvals}(D) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\text{eigenvals}(C) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The eigenvectors of the operators: $\text{eigenvecs}(D) = \begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix}$ $\text{eigenvecs}(C) = \begin{pmatrix} -0.707i & 0.707 \\ 0.707 & -0.707i \end{pmatrix}$

The following operators represent the measurement protocols for spins 1, 2 and 3. For example, the first operator designates that diagonal polarization is measured on the first photon and circular polarization on the second and third photons.

$D_1 \cdot C_2 \cdot C_3$ $C_1 \cdot D_2 \cdot C_3$ $C_1 \cdot C_2 \cdot D_3$ $D_1 \cdot D_2 \cdot D_3$

The four operators are constructed in matrix format using tensor multiplication.

$DCC := \text{kronecker}(D, \text{kronecker}(C, C))$ $CDC := \text{kronecker}(C, \text{kronecker}(D, C))$

$CCD := \text{kronecker}(C, \text{kronecker}(C, D))$ $DDD := \text{kronecker}(D, \text{kronecker}(D, D))$

1. The operators are unitary. Demonstrate this for DDC and DDD.

$$DCC^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad DDD^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. The operators are also Hermitian. Demonstrate this for CDC. What is the significance of being Hermitian?

$$CDC = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad CDC^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hermitian operators have real eigenvalues.

3. The operators have eigenvalues +/-1 and the same set of eigenvectors. Demonstrate this for DCC and CCD.

$$\text{eigenvals}(DCC) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{eigenvecs}(DCC) = \begin{pmatrix} 0.707 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0.707 \\ 0 & 0 & 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.707 & -0.707 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.707 & -0.707 & 0 & 0 \\ 0 & 0 & 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & -0.707 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{eigenvals}(CCD) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{eigenvecs}(CCD) = \begin{pmatrix} 0.707 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 0.707 \\ 0 & 0 & 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.707 & -0.707 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0.707 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. One of the eigenvectors is Ψ. Calculate its expectation values for the four operators.

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi^T \cdot DCC \cdot \Psi = 1 \qquad \Psi^T \cdot CDC \cdot \Psi = -1 \qquad \Psi^T \cdot CCD \cdot \Psi = -1 \qquad \Psi^T \cdot DDD \cdot \Psi = -1$$

$$\Psi^T \cdot DCC \cdot CDC \cdot CCD \cdot \Psi = 1$$

5. Demonstrate that Ψ is a maximally entangle three-photon state. These are called GHZ states.

$$\Psi = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \frac{1}{\sqrt{2}}(3 - 4) \qquad \text{in decimal notation}$$

6. Demonstrate that these operators mutually commute (use the symbolic processor --->). What is the physical significance of this fact?

$$DCC \cdot CDC - CDC \cdot DCC \rightarrow 0 \qquad DCC \cdot CCD - CCD \cdot DCC \rightarrow 0 \qquad DCC \cdot DDD - DDD \cdot DCC \rightarrow 0$$

$$CDC \cdot CCD - CCD \cdot CDC \rightarrow 0 \qquad CDC \cdot DDD - DDD \cdot CDC \rightarrow 0 \qquad CCD \cdot DDD - DDD \cdot CCD \rightarrow 0$$

That these operators mutually commute means that they can have simultaneous eigenstates with simultaneous eigenvalues. In other words, they can be in well-defined states at the same time.

7. Display $DCC \cdot CDC \cdot CCD$ and DDD in matrix format (use the traditional =).

$$DCC \cdot CDC \cdot CCD = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad DDD = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

8. Write this result in simple algebraic format.

$$DCC \cdot CDC \cdot CCD = -DDD$$

9. Using arguments based on local realism demonstrate that this result leads to the following contradiction.

$$DDD = -DDD$$

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes the diagonal and circular polarization states have definite values prior to measurement. This position leads to a contradiction with the above result. There is no way to assign eigenvalues (+/-1) to the operators that is consistent with the above result.

$$DCC \cdot CDC \cdot CCD = -DDD$$

Concentrating on the composite operator on the left side, we notice that there is a **C** measurement on the first photon in the second and third operators (green). If the photon state is well-defined before measurement those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1. There is a **C** measurement on the second photon in operators one and three (blue). By similar arguments those results will lead to a product of +1 also. Finally there is a **C** measurement on the third photon in operators one and two (red). By similar arguments those results will lead to a product of +1 also. Incorporating these observations into the expression above leads to the following contradiction.

$$DDD = -DDD$$

This result should cause all mathematically literate local realists to renounce and recant their heresy.