Another Simulation of a GHZ Gedanken Experiment

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Twenty years ago N. David Mermin published two articles (*Physics Today*, June 1990; *American Journal of Physics*, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (*American Journal of Physics*, December 1990; *Nature*, 3 February 2000) thought experiment involving spins that sharply revealed the clash between local realism and the quantum view of reality.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of h/4π with spin operators in the x- and y-directions.

The z-basis eigenfunctions are: $Sz_{up} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad Sz_{down} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The x- and y-direction spin operators : $\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \text{eigenvals}(\sigma_x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \text{eigenvals}(\sigma_y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The initial entangled spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The following operators represent the measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \qquad \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The four operators in tensor format are formed as follows.

$$\sigma_{xyy} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_y, \sigma_y)) \qquad \sigma_{yxy} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_x, \sigma_y))$$

$$\sigma_{yyx} := \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_y, \sigma_x)) \qquad \sigma_{xxx} := \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_x, \sigma_x))$$

These composite operators are Hermitian and mutually commute which means they can have simultaneous eigenvalues.

$$\sigma_{xyy} \cdot \sigma_{yxy} - \sigma_{yxy} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{xyy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{xyy} \rightarrow 0 \qquad \sigma_{xyy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{xyy} \rightarrow 0$$

$$\sigma_{yxy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{yxy} \rightarrow 0 \qquad \sigma_{yxy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yxy} \rightarrow 0 \qquad \sigma_{yyx} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yyx} \rightarrow 0$$

The expectation values of the operators are now calculated.

$$\Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1 \qquad \Psi^T \cdot \sigma_{yxy} \cdot \Psi = 1 \qquad \Psi^T \cdot \sigma_{yyx} \cdot \Psi = 1 \qquad \Psi^T \cdot \sigma_{xxx} \cdot \Psi = -1$$

Consequently the product of the four operators has the expectation value of -1.

$$\Psi^T \cdot \sigma_{xyy} \cdot \sigma_{yxy} \cdot \sigma_{yyx} \cdot \sigma_{xxx} \cdot \Psi = -1$$

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes that the x- and y-components of the spin have definite values prior to measurement. This position leads to a contradiction with the above result as demonstrated by Mermin (*Physics Today*, June 1990). Looking again at the measurement operators, notice that there is a σ_x measurement on the first spin in the first and fourth experiment. If the spin state is well-defined before measurement those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1.

$$\left(\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3\right)\left(\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3\right)\left(\sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3\right)\left(\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3\right)$$

Likewise there is a σ_y measurement on the second spin in experiments one and three. By similar arguments those results will lead to a product of +1 also. Continuing with all pairs in the total operator using local realistic reasoning unambiguously shows that its expectation value should be +1, in sharp disagreement with the quantum mechanical result of -1. **This result should cause all mathematically literate local realists to renounce and recant their heresy.** However, they may resist saying this is just a thought experiment. It hasn't actually been performed. However, if you believe in quantum simulation it has been performed.

Quantum Simulation

"Quantum simulation is a process in which a quantum computer simulates another quantum system. Because of the various types of quantum weirdness, classical computers can simulate quantum systems only in a clunky, inefficient way. But because a quantum computer is itself a quantum system, capable of exhibiting the full repertoire of quantum weirdness, it can efficiently simulate other quantum systems. **The resulting simulation can be so accurate that the behavior the computer will be indistinguishable from the behavior of the simulated system itself.**" (Seth Lloyd, Programming the Universe, page 149.)

The thought experiment can be simulated using the quantum circuit shown below which is an adaptation of one that can be found at: arXiv:1712.06542v2.

```
|0⟩ ▷  H   •   •   S   ···  H  ▷  Measure, 0 or 1
           |   |
|0⟩ ▷ ···  ⊕   |  ···  S'   H  ▷  Measure, 0 or 1
               |
|0⟩ ▷ ··· ···  ⊕  ···  S'   H  ▷  Measure, 0 or 1
```

The matrix operators required for the implementation of the quantum circuit:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S' := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad CCNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$HII := kronecker(H, kronecker(I, I))$ $CNOTI := kronecker(CNOT, I)$ $SII := kronecker(S, kronecker(I, I))$

$IS'S' := kronecker(I, kronecker(S', S'))$ $S'IS' := kronecker(S', kronecker(I, S'))$ $S'S'I := kronecker(S', kronecker(S', I))$

$HHH := kronecker(H, kronecker(H, H))$

First it is demonstrated that the first four steps of the circuit create the initial state.

$$\left[SII \cdot CCNOT \cdot CNOTI \cdot HII \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T\right]^T = (0.707\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.707)$$

The complete circuit shown above simulates the expectation value of the $\sigma_x\sigma_y\sigma_y$ operator. The presence of S' on a line before the final H gates indicates the measurement of the σ_y, its absence a measurement of σ_x. The subsequent simulations show the absence of S' on the middle and last line, and finally on all three lines for the simulation of the expectation value for $\sigma_x\sigma_x\sigma_x$.

Eigenvalue |0> = +1; eigenvalue |1> = -1

$$\left[HHH \cdot IS'S' \cdot SII \cdot CCNOT \cdot CNOTI \cdot HII \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T\right]^T = (0.5\ \ 0\ \ 0\ \ 0.5\ \ 0\ \ 0.5\ \ 0.5\ \ 0)$$

$$\frac{1}{2}\left(\left|000\right\rangle + \left|011\right\rangle + \left|101\right\rangle + \left|110\right\rangle\right) \Rightarrow \left\langle \sigma_x\sigma_y\sigma_y \right\rangle = 1$$

Given the eigenvalue assignments above the expectation value associated with this measurement outcome is 1/4[(1)(1)(1)+(1)(-1)(-1)+(-1)(1)(-1)+(-1)(-1)(1)] = 1. Note that 1/2 is the probability amplitude for the product state. Therefore the probability of each member of the superposition being observed is 1/4. The same reasoning is used for the remaining simulations.

$$\left[HHH \cdot S'IS' \cdot SII \cdot CCNOT \cdot CNOTI \cdot HII \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T\right]^T = (0.5\ \ 0\ \ 0\ \ 0.5\ \ 0\ \ 0.5\ \ 0.5\ \ 0)$$

$$\frac{1}{2}\left(\left|000\right\rangle + \left|011\right\rangle + \left|101\right\rangle + \left|110\right\rangle\right) \Rightarrow \left\langle \sigma_y\sigma_x\sigma_y \right\rangle = 1$$

$$\left[HHH \cdot S'S'I \cdot SII \cdot CCNOT \cdot CNOTI \cdot HII \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T\right]^T = (0.5\ \ 0\ \ 0\ \ 0.5\ \ 0\ \ 0.5\ \ 0.5\ \ 0)$$

$$\frac{1}{2}\left(\left|000\right\rangle + \left|011\right\rangle + \left|101\right\rangle + \left|110\right\rangle\right) \Rightarrow \left\langle \sigma_y\sigma_y\sigma_x \right\rangle = 1$$

$$\left[HHH \cdot SII \cdot CCNOT \cdot CNOTI \cdot HII \cdot (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T\right]^T = (0\ \ 0.5\ \ 0.5\ \ 0\ \ 0.5\ \ 0\ \ 0\ \ 0.5)$$

$$\frac{1}{2}\left(\left|001\right\rangle + \left|010\right\rangle + \left|100\right\rangle + \left|111\right\rangle\right) \Rightarrow \left\langle \sigma_x\sigma_x\sigma_x \right\rangle = -1$$

Individually and in product form the simulated results are in agreement with the previous quantum mechanical calculations.

$$\left\langle \sigma_x\sigma_x\sigma_x \right\rangle \left\langle \sigma_x\sigma_y\sigma_y \right\rangle \left\langle \sigma_y\sigma_x\sigma_y \right\rangle \left\langle \sigma_y\sigma_y\sigma_x \right\rangle = -1$$

The same simulations can be carried out using the Quirk Quantum Simulator (algassert.com/quirk). The following circuits simulate the $\sigma_x\sigma_y\sigma_y$ and $\sigma_x\sigma_x\sigma_x$ expectation values which are +1 and -1. The first circuit is easily modified for the $\sigma_y\sigma_x\sigma_y$ and $\sigma_y\sigma_y\sigma_x$ simulations.

$$\begin{array}{cccccccccc}
|1\rangle & \triangleright & H & \cdot & & & X & \triangleright & \\
 & & & | & & & & & \\
|0\rangle & \triangleright & \cdots & \oplus & \cdot & \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right) & Y & \triangleright & \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right) \\
 & & & & | & & & & \\
|0\rangle & \triangleright & & & \oplus & & Y & \triangleright &
\end{array}$$

$$\begin{array}{cccccccccc}
|1\rangle & \triangleright & H & \cdot & & & X & \triangleright & \\
 & & & | & & & & & \\
|0\rangle & \triangleright & \cdots & \oplus & \cdot & \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right) & X & \triangleright & -\frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right) \\
 & & & & | & & & & \\
|0\rangle & \triangleright & & & \oplus & & X & \triangleright &
\end{array}$$