Grover's Search Algrorithm: Four Card Monte

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Grover's search algorithm is great at playing four-card monte. As the following quantum circuit shows it can determine which card is the queen in one pass.

$$\begin{matrix} |0\rangle & H & \triangleright & \begin{bmatrix} \\ \text{Oracle} \\ \\ \end{bmatrix} & H & \begin{bmatrix} \\ J \\ \\ \end{bmatrix} & H & \triangleright & \text{Measure} \\ |0\rangle & H & \triangleright & & H & & H & \triangleright & \text{Measure} \end{matrix}$$

where

$$\begin{bmatrix} \\ J \\ \\ \end{bmatrix} = \begin{matrix} X & \bullet & X \\ & | & \\ X & \boxed{Z} & X \end{matrix}$$

The following matrix operators are required to construct the circuit. Giving |10> a negative phase in the Oracle designates it as the queen. The Appendix shows the calculation of J as shown on the right side above.

$$H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad HH := \text{kronecker}(H, H) \qquad \text{Oracle} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad J := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operating on the input state, which creates a superposition of all queries, enables the algorithm to identify which card is the queen in one operation of the circuit.

$$\text{GroverSearch} := HH \cdot J \cdot HH \cdot \text{Oracle} \cdot HH \qquad \text{GroverSearch} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$ = -|10>

Now the operation of the algorithm is carried out in steps to show the importance of constructive and destructive interference in the execution of the quantum circuit.

Step 1 $$HH \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$ = 1/2[|00> + |01> + |10> + |11>]

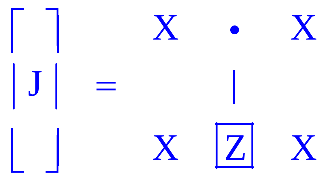
Step 2 $$\text{Oracle} \cdot \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{pmatrix}$$ = 1/2[|00> + |01> - |10> + |11>]

Step 3 $$HH \cdot \begin{pmatrix} 0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$ = 1/2[|00> - |01> + |10> + |11>]

Step 4 $$J \cdot \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$ = 1/2[-|00> - |01> + |10> + |11>]

Step 5 $$HH\cdot\begin{pmatrix}-0.5\\-0.5\\0.5\\0.5\end{pmatrix}=\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}$$ = -|10>

Appendix



$$J=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad X:=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad CZ:=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\qquad \text{kronecker}(X,X)\cdot CZ\cdot\text{kronecker}(X,X)=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

An algebraic summary of the search algorithm:

$$|0\rangle|0\rangle$$

$$H\otimes H$$

$$\tfrac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\tfrac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)=\tfrac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)$$

$$\text{Oracle}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

$$\tfrac{1}{2}\left(|00\rangle+|01\rangle-|10\rangle+|11\rangle\right)$$

$$H\otimes H$$

$$\tfrac{1}{2}\left(|00\rangle-|01\rangle+|10\rangle+|11\rangle\right)$$

$$J=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\tfrac{1}{2}\left(-|00\rangle-|01\rangle+|10\rangle+|11\rangle\right)$$

$$H\otimes H$$

$$-|10\rangle$$