An Entanglement Swapping Protocol

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In the field of quantum information interference, superpositions and entangled states are essential resources. Entanglement, a non-factorable superposition, is routinely achieved when two photons are emitted from the same source, say a parametric down converter (PDC). Entanglement swapping involves the transfer of entanglement to two photons that were produced independently and never previously interacted. The Bell states are the four maximally entangled two-qubit entangled basis for a four-dimensional Hilbert space and play an essential role in quantum information theory and technology, including teleportation and entanglement swapping. The Bell states are shown below.

$$\Phi_p = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad \Phi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \qquad \Phi_m = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad \Phi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\Psi_p = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad \Psi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \qquad \Psi_m = \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right] \qquad \Psi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

A four-qubit state is prepared in which photons 1 and 2 are entangled in Bell state Φ_p, and photons 3 and 4 are entangled in Bell state Ψ_m. The state multiplication below is understood to be tensor vector multiplication.

$$\Psi = \Phi_p\cdot\Psi_m = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \qquad \Psi := \frac{1}{2}\cdot\begin{pmatrix}0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\end{pmatrix}^T \qquad I := \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Four Bell state measurements are now made on photons 2 and 3 which entangles photons 1 and 4.

Projection of photons 2 and 3 onto Φ_p projects photons 1 and 4 onto Ψ_m.

$$\left(\text{kronecker}\left(I,\text{kronecker}\left(\Phi_p\cdot\Phi_p{}^T,I\right)\right)\cdot\Psi\right)^T = \begin{pmatrix}0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & -0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0\end{pmatrix}$$

$$\left[\frac{1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]\right]^T = \frac{1}{4}\cdot\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\end{pmatrix}$$

Projection of photons 2 and 3 onto Φ_m projects photons 1 and 4 onto Ψ_p.

$$\left(\text{kronecker}\left(I,\text{kronecker}\left(\Phi_m\cdot\Phi_m{}^T,I\right)\right)\cdot\Psi\right)^T = \begin{pmatrix}0 & 0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0\end{pmatrix}$$

$$\left[\frac{1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]\right]^T = \frac{1}{4}\cdot\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\end{pmatrix}$$

Projection of photons 2 and 3 onto Ψ_p projects photons 1 and 4 onto Φ_m.

$$\left(\text{kronecker}\left(I, \text{kronecker}\left(\Psi_p \cdot \Psi_p^{\,T}, I\right)\right) \cdot \Psi\right)^T = (0\ \ 0\ \ -0.25\ \ 0\ \ -0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.25\ \ 0\ \ 0.25\ \ 0\ \ 0)$$

$$\frac{1}{2\cdot\sqrt{2}} \cdot \left[\begin{pmatrix}0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix}\right]^T = \frac{1}{4} \cdot (0\ \ 0\ \ -1\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0)$$

Finally, projection of photons 2 and 3 onto Ψ_m projects photons 1 and 4 onto Φ_p.

$$\left(\text{kronecker}\left(I, \text{kronecker}\left(\Psi_m \cdot \Psi_m^{\,T}, I\right)\right) \cdot \Psi\right)^T = (0\ \ 0\ \ -0.25\ \ 0\ \ 0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.25\ \ 0\ \ 0.25\ \ 0\ \ 0)$$

$$\frac{-1}{2\cdot\sqrt{2}} \cdot \left[\begin{pmatrix}1\\0\end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix}\right]^T = \frac{1}{4} \cdot (0\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 0)$$