Entanglement Swapping

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In the field of quantum information, interference, superpositions and entangled states are essential resources. Entanglement, a non-factorable superposition, is routinely achieved when two photons are emitted from the same source, perhaps a parametric down converter (PDC). Entanglement swapping involves the transfer (teleportation) of entanglement to two photons that were produced independently and never previously interacted. The Bell states are the four maximally entangled two-qubit entangled basis for a four-dimensional Hilbert space and play an essential role in quantum information theory and technology, including teleportation and entanglement swapping. This analysis attempts to provide the essential matrix math needed to understand parts of "Entangled delayed-choice entanglement swapping" (arXiv1203.4834) and "Delayed-choice gedanken experiments and their realizations" (arXiv1407.2930). The following analysis deals exclusively with entanglement swapping and does not consider the delayed-choice aspect of the research presented in these papers.

Bell states and the identity operator:

$$\Phi_p := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \Phi_m := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \Psi_p := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \Psi_m := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A four-qubit state is prepared in which photons 1 and 2 are entangled in Bell state Ψ_m, and photons 3 and 4 are also entangled in Bell state Ψ_m. The state multiplication below is understood to be tensor vector multiplication.

$$\Psi_{1234} = \Psi_{m_{12}} \cdot \Psi_{m_{34}} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \Psi := \frac{1}{2} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

The authors write this state as a superposition of Bell state products to suggest a way to transfer entanglement from 1&2-3&4 to 1&4-2&3: perform a Bell state measurement on photons 2 and 3.

$$\left|\Psi\right\rangle_{1234} = \frac{1}{2}\left[\left|\Psi_p\right\rangle_{14} \otimes \left|\Psi_p\right\rangle_{23} - \left|\Psi_m\right\rangle_{14} \otimes \left|\Psi_m\right\rangle_{23} - \left|\Phi_p\right\rangle_{14} \otimes \left|\Phi_p\right\rangle_{23} + \left|\Phi_m\right\rangle_{14} \otimes \left|\Phi_m\right\rangle_{23}\right]$$

The following calculations agree with this product of entangled photon pairs 1&4 and 2&3.

Projection of photons 2 and 3 onto Ψ_p projects photons 1 and 4 onto Ψ_p.

$$\left(\text{kronecker}\left(I, \text{kronecker}\left(\Psi_p \cdot \Psi_p^T, I\right)\right) \cdot \Psi\right)^T = \begin{pmatrix} 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{2 \cdot \sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^T = \frac{1}{4} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Projection of photons 2 and 3 onto Ψ_m projects photons 1 and 4 onto -Ψ_m.

$$\left(\text{kronecker}\left(I, \text{kronecker}\left(\Psi_m \cdot \Psi_m^T, I\right)\right) \cdot \Psi\right)^T = \begin{pmatrix} 0 & 0 & 0 & -0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0.25 & 0 & -0.25 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{-1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]^T = \frac{1}{4}\cdot(0\ \ 0\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0)$$

Projection of photons 2 and 3 onto Φ_p projects photons 1 and 4 onto $-\Phi_p$.

$$\left(\text{kronecker}\left(I,\text{kronecker}\left(\Phi_p\cdot\Phi_p^T,I\right)\right)\cdot\Psi\right)^T = (-0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.25\ \ 0\ \ 0\ \ -0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.25)$$

$$\frac{-1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right]^T = \frac{1}{4}\cdot(-1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1\ \ 0\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1)$$

Projection of photons 2 and 3 onto Φ_m projects photons 1 and 4 onto Φ_m.

$$\left(\text{kronecker}\left(I,\text{kronecker}\left(\Phi_m\cdot\Phi_m^T,I\right)\right)\cdot\Psi\right)^T = (0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.25\ \ 0\ \ 0\ \ -0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.25)$$

$$\frac{1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right]^T = \frac{1}{4}\cdot(1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1\ \ 0\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1)$$

The initial four-particle state can be written in the H/V, A/D and R/L polarization bases. See the Appendix for details.

$$\Psi_{1234} = \frac{1}{2}\cdot\left(H_1\cdot V_2 - V_1\cdot H_2\right)\cdot\left(H_3\cdot V_4 - V_3\cdot H_4\right) = \frac{1}{2}\cdot\left(H_1\cdot V_2\cdot H_3\cdot V_4 - H_1\cdot V_2\cdot V_3\cdot H_4 - V_1\cdot H_2\cdot H_3\cdot V_4 + V_1\cdot H_2\cdot V_3\cdot H_4\right)$$

$$\Psi_{1234} = \frac{1}{2}\cdot\left(A_1\cdot D_2 - D_1\cdot A_2\right)\cdot\left(A_3\cdot D_4 - D_3\cdot A_4\right) = \frac{1}{2}\cdot\left(A_1\cdot D_2\cdot A_3\cdot D_4 - A_1\cdot D_2\cdot D_3\cdot A_4 - D_1\cdot A_2\cdot A_3\cdot D_4 + D_1\cdot A_2\cdot D_3\cdot A_4\right)$$

$$\Psi_{1234} = \frac{1}{2}\cdot\left(L_1\cdot R_2 - R_1\cdot L_2\right)\cdot\left(L_3\cdot R_4 - R_3\cdot L_4\right) = \frac{1}{2}\cdot\left(L_1\cdot R_2\cdot L_3\cdot R_4 - L_1\cdot R_2\cdot R_3\cdot L_4 - R_1\cdot L_2\cdot L_3\cdot R_4 + R_1\cdot L_2\cdot R_3\cdot L_4\right)$$

Where,

$$H := \begin{pmatrix}1\\0\end{pmatrix} \qquad V := \begin{pmatrix}0\\1\end{pmatrix} \qquad R := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\i\end{pmatrix} \qquad L := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-i\end{pmatrix} \qquad D := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\1\end{pmatrix} \qquad A := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-1\end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\cdot(R + L) \qquad V = \frac{i}{\sqrt{2}}\cdot(L - R) \qquad H = \frac{1}{\sqrt{2}}\cdot(D + A) \qquad V = \frac{1}{\sqrt{2}}\cdot(D - A)$$

$$R = \frac{1}{\sqrt{2}}\cdot(H + i\cdot V) \qquad L = \frac{1}{\sqrt{2}}\cdot(H - i\cdot V) \qquad R = \frac{1}{\sqrt{2}}\cdot(D + A)$$

After production, photon 1 is sent to Alice and photon 4 is sent to Bob. Photons 2 and 3 are sent to Victor. Alice and Bob can measure their photons in either the H/V, R/L or A/D bases. Victor can measure his photons separately in the H/V basis or he can carry out one of the four Bell state measurements on his photon pairs. This later choice as shown earlier projects photons 1 and 4 into the corresponding entangled Bell state.

The following table shows the possible measurement results that Alice and Bob will obtain, depending on the type of measurement Victor makes on his photons. In experiments 1-4 Victor measures his photons separately in the H/V basis and Alice and Bob do the same. In the remaining experiments Victor does a Bell state measurement on his photons and Alice and Bob measure in any of the three bases.

Experiment	1	2	3	4	'	5	6	7	8	'	9	10	11	12	'	13	14	15	16
Alice1	H	H	V	V	'	H	V	H	V	'	L	R	L	R	'	D	A	A	D
Bob4	V	H	V	H	'	V	H	H	V	'	L	L	R	R	'	D	D	A	A
Victor23	VH	VV	HH	HV	'	Ψ_p	Ψ_m	Φ_p	Φ_m	'	Ψ_p	Ψ_m	Φ_p	Φ_m	'	Ψ_p	Ψ_m	Φ_p	Φ_m

Results 1-4 are consistent with the original state expressed in the H/V basis.

$$\Psi_{1234} = \frac{1}{2}\cdot\left(H_1\cdot V_2\cdot H_3\cdot V_4 - H_1\cdot V_2\cdot V_3\cdot H_4 - V_1\cdot H_2\cdot H_3\cdot V_4 + V_1\cdot H_2\cdot V_3\cdot H_4\right)$$

In the remaining experiments Victor makes a Bell state measurement on photons 2 and 3, and projects photons 1 and 4 into the following Bell states. The table shows measurement results that Alice and Bob could make on their photons given these states. See the Appendix for more detail

$$\Psi_{p_{14}} = \frac{1}{\sqrt{2}}\cdot\left(H_1\cdot V_4 + V_1\cdot H_4\right) = \frac{i}{\sqrt{2}}\cdot\left(L_1\cdot L_4 - R_1\cdot R_4\right) = \frac{1}{\sqrt{2}}\cdot\left(D_1\cdot D_4 - A_1\cdot A_4\right)$$

$$\Psi_{m_{14}} = \frac{1}{\sqrt{2}}\cdot\left(H_1\cdot V_4 - V_1\cdot H_4\right) = \frac{i}{\sqrt{2}}\cdot\left(L_1\cdot R_4 - R_1\cdot L_4\right) = \frac{1}{\sqrt{2}}\cdot\left(A_1\cdot D_4 - D_1\cdot A_4\right)$$

$$\Phi_{p_{14}} = \frac{1}{\sqrt{2}}\cdot\left(H_1\cdot H_4 + V_1\cdot V_4\right) = \frac{1}{\sqrt{2}}\cdot\left(L_1\cdot R_4 + R_1\cdot L_4\right) = \frac{1}{\sqrt{2}}\cdot\left(A_1\cdot A_4 + D_1\cdot D_4\right)$$

$$\Phi_{m_{14}} = \frac{1}{\sqrt{2}}\cdot\left(H_1\cdot H_4 - V_1\cdot V_4\right) = \frac{1}{\sqrt{2}}\cdot\left(L_1\cdot L_4 + R_1\cdot R_4\right) = \frac{1}{\sqrt{2}}\cdot\left(A_1\cdot D_4 + D_1\cdot A_4\right)$$

Appendix

The Bell states are written in the H/V, R/L and A/D bases.

$$\Psi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot V_j + V_i\cdot H_j\right) \begin{vmatrix} \text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(R_i + L_i\right) \\ \text{substitute}, V_j = \frac{i}{\sqrt{2}}\cdot\left(L_j - R_j\right) \\ \text{substitute}, V_i = \frac{i}{\sqrt{2}}\cdot\left(L_i - R_i\right) \\ \text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(R_j + L_j\right) \end{matrix} \rightarrow \sqrt{2}\cdot\left(-\frac{R_i\cdot R_j\cdot i}{2} + \frac{L_i\cdot L_j\cdot i}{2}\right)$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot V_j + V_i\cdot H_j\right) \left| \begin{array}{l} \text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(D_i + A_i\right) \\ \text{substitute}, V_j = \frac{1}{\sqrt{2}}\cdot\left(D_j - A_j\right) \\ \text{substitute}, V_i = \frac{1}{\sqrt{2}}\cdot\left(D_i - A_i\right) \\ \text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(D_j + A_j\right) \end{array} \right. \rightarrow -\sqrt{2}\cdot\left(\frac{A_i\cdot A_j}{2} - \frac{D_i\cdot D_j}{2}\right)$$

$$\Psi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot V_j - V_i\cdot H_j\right) \left| \begin{array}{l} \text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(R_i + L_i\right) \\ \text{substitute}, V_j = \frac{i}{\sqrt{2}}\cdot\left(L_j - R_j\right) \\ \text{substitute}, V_i = \frac{i}{\sqrt{2}}\cdot\left(L_i - R_i\right) \\ \text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(R_j + L_j\right) \end{array} \right. \rightarrow -\sqrt{2}\cdot\left(-\frac{L_j\cdot R_i\cdot i}{2} + \frac{L_i\cdot R_j\cdot i}{2}\right)$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot V_j - V_i\cdot H_j\right) \left| \begin{array}{l} \text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(D_i + A_i\right) \\ \text{substitute}, V_j = \frac{1}{\sqrt{2}}\cdot\left(D_j - A_j\right) \\ \text{substitute}, V_i = \frac{1}{\sqrt{2}}\cdot\left(D_i - A_i\right) \\ \text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(D_j + A_j\right) \end{array} \right. \rightarrow \sqrt{2}\cdot\left(\frac{A_i\cdot D_j}{2} - \frac{A_j\cdot D_i}{2}\right)$$

$$\Phi_p := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot H_j + V_i\cdot V_j\right) \left| \begin{array}{l} \text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(R_i + L_i\right) \\ \text{substitute}, V_j = \frac{i}{\sqrt{2}}\cdot\left(L_j - R_j\right) \\ \text{substitute}, V_i = \frac{i}{\sqrt{2}}\cdot\left(L_i - R_i\right) \\ \text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(R_j + L_j\right) \end{array} \right. \rightarrow \sqrt{2}\cdot\left(\frac{L_i\cdot R_j}{2} + \frac{L_j\cdot R_i}{2}\right)$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot H_j + V_i\cdot V_j\right) \left| \begin{array}{l} \text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(D_i + A_i\right) \\ \text{substitute}, V_j = \frac{1}{\sqrt{2}}\cdot\left(D_j - A_j\right) \\ \text{substitute}, V_i = \frac{1}{\sqrt{2}}\cdot\left(D_i - A_i\right) \\ \text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(D_j + A_j\right) \end{array} \right. \rightarrow \sqrt{2}\cdot\left(\frac{A_i\cdot A_j}{2} + \frac{D_i\cdot D_j}{2}\right)$$

$$\Phi_m := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot H_j - V_i\cdot V_j\right) \left|\begin{aligned}&\text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(R_i + L_i\right)\\&\text{substitute}, V_j = \frac{i}{\sqrt{2}}\cdot\left(L_j - R_j\right)\\&\text{substitute}, V_i = \frac{i}{\sqrt{2}}\cdot\left(L_i - R_i\right)\\&\text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(R_j + L_j\right)\end{aligned}\right. \rightarrow \sqrt{2}\cdot\left(\frac{L_i\cdot L_j}{2} + \frac{R_i\cdot R_j}{2}\right)$$

$$\frac{1}{\sqrt{2}}\cdot\left(H_i\cdot H_j - V_i\cdot V_j\right) \left|\begin{aligned}&\text{substitute}, H_i = \frac{1}{\sqrt{2}}\cdot\left(D_i + A_i\right)\\&\text{substitute}, V_j = \frac{1}{\sqrt{2}}\cdot\left(D_j - A_j\right)\\&\text{substitute}, V_i = \frac{1}{\sqrt{2}}\cdot\left(D_i - A_i\right)\\&\text{substitute}, H_j = \frac{1}{\sqrt{2}}\cdot\left(D_j + A_j\right)\end{aligned}\right. \rightarrow \sqrt{2}\cdot\left(\frac{A_i\cdot D_j}{2} + \frac{A_j\cdot D_i}{2}\right)$$

The initial state is written in the H/V, R/L and A/D bases.

$$\frac{1}{2}\cdot\left(H_1\cdot V_2 - V_1\cdot H_2\right)\cdot\left(H_3\cdot V_4 - V_3\cdot H_4\right) \left|\begin{aligned}&\text{substitute}, H_1 = \frac{1}{\sqrt{2}}\cdot\left(D_1 + A_1\right)\\&\text{substitute}, V_2 = \frac{1}{\sqrt{2}}\cdot\left(D_2 - A_2\right)\\&\text{substitute}, V_1 = \frac{1}{\sqrt{2}}\cdot\left(D_1 - A_1\right)\\&\text{substitute}, H_2 = \frac{1}{\sqrt{2}}\cdot\left(D_2 + A_2\right)\\&\text{substitute}, H_3 = \frac{1}{\sqrt{2}}\cdot\left(D_3 + A_3\right)\\&\text{substitute}, V_3 = \frac{1}{\sqrt{2}}\cdot\left(D_3 - A_3\right)\\&\text{substitute}, H_4 = \frac{1}{\sqrt{2}}\cdot\left(D_4 + A_4\right)\\&\text{substitute}, V_4 = \frac{1}{\sqrt{2}}\cdot\left(D_4 - A_4\right)\end{aligned}\right. \rightarrow \frac{\left(A_1\cdot D_2 - A_2\cdot D_1\right)\cdot\left(A_3\cdot D_4 - A_4\cdot D_3\right)}{2}$$

$$\frac{1}{2}\cdot\left(H_1\cdot V_2 - V_1\cdot H_2\right)\cdot\left(H_3\cdot V_4 - V_3\cdot H_4\right) \left| \begin{array}{l} \text{substitute}, H_1 = \frac{1}{\sqrt{2}}\cdot\left(R_1 + L_1\right) \\ \text{substitute}, V_2 = \frac{i}{\sqrt{2}}\cdot\left(L_2 - R_2\right) \\ \text{substitute}, V_1 = \frac{i}{\sqrt{2}}\cdot\left(L_1 - R_1\right) \\ \text{substitute}, H_2 = \frac{1}{\sqrt{2}}\cdot\left(R_2 + L_2\right) \\ \text{substitute}, H_3 = \frac{1}{\sqrt{2}}\cdot\left(R_3 + L_3\right) \\ \text{substitute}, V_3 = \frac{i}{\sqrt{2}}\cdot\left(L_3 - R_3\right) \\ \text{substitute}, H_4 = \frac{1}{\sqrt{2}}\cdot\left(R_4 + L_4\right) \\ \text{substitute}, V_4 = \frac{i}{\sqrt{2}}\cdot\left(L_4 - R_4\right) \end{array} \right. \rightarrow -\frac{\left(L_1\cdot R_2 - L_2\cdot R_1\right)\cdot\left(L_3\cdot R_4 - L_4\cdot R_3\right)}{2}$$