Entanglement Reveals a Conflict Between Local Realism and Quantum Theory

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A tensor algebra approach is used to demonstrate the challenge to the local realistic position of reality that quantum mechanical entanglement creates. The example is drawn from Chapter 3 of David Z Albert's text, *Quantum Mechanics and Experience*.

A quon (any entity that exhibits both wave and particle aspects in the peculiar quantum manner - Nick Herbert, *Quantum Reality*, page 64) has a variety of properties each of which can take on two values. For example, it has the property of *hardness* and can be either *hard* or *soft*. It also has the property of *color* and can be either *black* or *white*.

In the matrix formulation of quantum mechanics these states are represented by the following vectors.

$$\text{Hard} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{Soft} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{Black} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{White} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Hard and *Soft* represent an orthonormal basis in the two-dimensional *Hardness* vector space.

$$\text{Hard}^T \cdot \text{Hard} = 1 \qquad \text{Soft}^T \cdot \text{Soft} = 1 \qquad \text{Hard}^T \cdot \text{Soft} = 0$$

Likewise *Black* and *White* are an orthonormal basis in the two-dimensional *Color* vector space.

$$\text{Black}^T \cdot \text{Black} = 1 \qquad \text{White}^T \cdot \text{White} = 1 \qquad \text{Black}^T \cdot \text{White} = 0$$

The relationship between the two bases is reflected in the following projection calculations.

$$\text{Hard}^T \cdot \text{Black} = 0.707 \qquad \text{Hard}^T \cdot \text{White} = 0.707 \qquad \text{Soft}^T \cdot \text{Black} = 0.707 \qquad \text{Soft}^T \cdot \text{White} = -0.707 \qquad \frac{1}{\sqrt{2}} = 0.707$$

Clearly *Black* and *White* can be written as superpositions of *Hard* and *Soft*, and vice versa.

$$\frac{1}{\sqrt{2}} \cdot (\text{Hard} + \text{Soft}) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (\text{Hard} - \text{Soft}) = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot (\text{Black} + \text{White}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (\text{Black} - \text{White}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hard, *Soft*, *Black* and *White* are measurable properties and the vectors representing them are eigenstates of the *Hardness* and *Color* operators with eigenvalues +/- 1.

Operators

$$\text{Hardness} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{Color} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalue +1 Eigenvalue -1

$$\text{Hardness}\cdot\text{Hard} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{Hardness}\cdot\text{Soft} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\text{Color}\cdot\text{Black} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \text{Color}\cdot\text{White} = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$$

Hard and *Soft* are not eigenfunctions of the *Color* operator, and *Black* and *White* are not eigenfunctions of the *Hardness* operator.

$$\text{Hardness}\cdot\text{Black} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \qquad \text{Hardness}\cdot\text{White} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

$$\text{Color}\cdot\text{Hard} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{Color}\cdot\text{Soft} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

As the *Hardness-Color* commutator shows, the *Hardness* and *Color* operators do not commute. They represent incompatible observables; observables that cannot simultaneously have well-defined values.

$$\text{Hardness}\cdot\text{Color} - \text{Color}\cdot\text{Hardness} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

We now proceed with an analysis of the implications of the following two-quon entangled state, expressed in tensor format. A pair of quons is prepared in the following "singlet" state; one is hard and one is soft. (The Appendix shows how to set this state up using Mathcad.)

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\text{Hard}\right\rangle_1\left|\text{Soft}\right\rangle_2 - \left|\text{Soft}\right\rangle_1\left|\text{Hard}\right\rangle_2\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$\Psi := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Given |Ψ> the expectation value for measuring *Hardness* on the first quon is 0. The same is true for the second quon. In other words, it is equally likely for either quon to be *Hard* or *Soft*. (*Kronecker* is Mathcad's command for tensor multiplication of matrices. See the Appendix for more detail.)

$$\Psi^T\cdot\text{kronecker}(\text{Hardness}, I)\cdot\Psi = 0 \qquad \Psi^T\cdot\text{kronecker}(I, \text{Hardness})\cdot\Psi = 0$$

However, if one quon is found to be *Hard* by measurement, the second will be measured *Soft*, and vice versa. In other words, there is perfect anti-correlation between the joint measurement of this property on the two quons.

$$\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Hardness}) \cdot \Psi = -1$$

Given |Ψ> the expectation value for measuring *Color* on the first quon is 0. The same is true for the second quon. In other words, it is equally likely for either quon to be *Black* or *White*.

$$\Psi^T \cdot \text{kronecker}(\text{Color}, I) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(I, \text{Color}) \cdot \Psi = 0$$

However, if one quon is found to be *Black* by measurement, the second will be measured *White* and vice versa. In other words, there is perfect anti-correlation between the joint measurement of this property on the two quons.

$$\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Color}) \cdot \Psi = -1$$

Furthermore, as the following calculations show, there is no correlation between the measurement outcomes on *Color* and *Hardness .*

$$\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Color}) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(\text{Color}, \text{Hardness}) \cdot \Psi = 0$$

As the foundation for their belief in local realism, Einstein, Podolsky and Rosen (*EPR*) defined the concept of *element of reality* in their famous 1935 *Physical Review* paper,

> "If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an *element of reality* corresponding to this physical quantity."

It would seem from the above results, namely these,

$$\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Color}) \cdot \Psi = -1 \qquad \Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Hardness}) \cdot \Psi = -1$$

that according to *EPR* both hardness and color are *elements of reality*. If the hardness of quon 1 is measured and found to be soft, we know without measurement (given the reliability of quantum mechanical predictions) that quon 2 is hard. Likewise, if the color of quon 2 is measured and found to be white, we know without measurement that quon 1 is black. On the basis of these calculations, the realist constructs the following table which assigns well-defined hardness and color states to both quons and is consistent with all the quantum calculations.

Quon1	Quon2	HardnessHardness	ColorColor	HardnessColor
HB	SW	−1	−1	−1
HW	SB	−1	−1	1
SB	HW	−1	−1	1
SW	HB	−1	−1	−1
Realist	AverageValue	−1	−1	0
Quantum	AverageValue	−1	−1	0

The problem with this interpretation is that it has previously been shown that the *Hardness* and *Color* operators do not commute, meaning that they represent incompatible observables. Incompatible observables cannot be known (determined) simultaneously. A contradiction between the EPR reality criterion and quantum mechanics has thus been shown to exist.

Appendix

Tensor multiplication is used to construct the initial state using Mathcad commands *submatrix, kronecker* and *augment.*

$$\Psi := \frac{1}{\sqrt{2}} \cdot \left[\text{submatrix}\left[\text{kronecker}\left[\text{augment}\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right],\text{augment}\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right]\right],1,4,1,1\right] \ldots \atop + -\text{submatrix}\left[\text{kronecker}\left[\text{augment}\left[\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right],\text{augment}\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right]\right],1,4,1,1\right] \right]$$

Kronecker is the Mathcad command that carries out the tensor multiplication of matrices. For example, consider the tensor multiplication of the *Hardness* and *Color* matrix operators.

$$Hardness \otimes Color = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} & 0\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\ 0\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} & -1\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & -1\\0 & 0 & -1 & 0\end{pmatrix}$$

$$\text{kronecker}(\text{Hardness}, \text{Color}) = \begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & -1\\0 & 0 & -1 & 0\end{pmatrix} \qquad \text{kronecker}\left[\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right] = \begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & -1\\0 & 0 & -1 & 0\end{pmatrix}$$