

Elements of Reality

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Twenty years ago N. David Mermin published two articles (*Physics Today*, June 1990; *American Journal of Physics*, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (*American Journal of Physics*, December 1990; *Nature*, 3 February 2000) gedanken experiment involving spins that sharply revealed the clash between local realism and the quantum view of reality. In what follows I present Mermin's *gedanken* experiment using tensor algebra.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of $\hbar/4\pi$ with spin operators in the x- and y-directions.

The z-basis eigenfunctions are: $Sz_{\text{up}} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $Sz_{\text{down}} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The x- and y-direction spin operators and eigenvalues:

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{eigenvals}(\sigma_x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{eigenvals}(\sigma_y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The initial spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \Psi := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The **Appendix** shows how to carry out vector tensor products in Mathcad.

The following operators represent the actual measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The three operators in tensor format are formed as follows.

$$\sigma_{xyy} := \text{kroncker}(\sigma_x, \text{kroncker}(\sigma_y, \sigma_y)) \quad \sigma_{xyy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{yxy} := \text{kroncker}(\sigma_y, \text{kroncker}(\sigma_x, \sigma_y)) \quad \sigma_{yxy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{yyx} := \text{kroncker}(\sigma_y, \text{kroncker}(\sigma_y, \sigma_x)) \quad \sigma_{yyx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

That the initial state is an eigenfunction of these operators with eigenvalue +1 is now demonstrated.

$$\Psi^T \cdot \sigma_{xyy} \cdot \Psi = 1$$

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The fact that the operators commute means that they can have simultaneous eigenvalues.

$$\sigma_{xyy} \cdot \sigma_{yxy} - \sigma_{yxy} \cdot \sigma_{xyy} \rightarrow 0$$

$$\sigma_{xyy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{xyy} \rightarrow 0$$

$$\sigma_{yxy} \cdot \sigma_{yyx} - \sigma_{yyx} \cdot \sigma_{yxy} \rightarrow 0$$

The significance of this is evident when we consider the eigenvalue of the product of the three operators which obviously must be +1.

$$(\sigma_x^1 \sigma_y^2 \sigma_y^3)(\sigma_y^1 \sigma_x^2 \sigma_y^3)(\sigma_y^1 \sigma_y^2 \sigma_x^3) = 1 \quad \Psi^T \cdot \sigma_{xyy} \cdot \sigma_{yxy} \cdot \sigma_{yyx} \cdot \Psi = 1$$

If it is assumed that this result occurs because the particles are in well-defined spin states (s_x, s_y) prior to measurement the following must be accepted,

$$\left(s_x^1 s_y^2 s_y^3\right)\left(s_y^1 s_x^2 s_y^3\right)\left(s_y^1 s_y^2 s_x^3\right)=1$$

Given that the spin measurement values can be +/- 1 and that the individual operators can have simultaneous eigenvalues, the following must be true,

$$s_y^1 s_y^1 = s_y^2 s_y^2 = s_y^3 s_y^3 = 1$$

This reduces the previous equation involving the three operators to the following local realistic prediction.

$$\left(s_x^1 s_x^2 s_x^3\right)=\left(\sigma_x^1 \sigma_x^2 \sigma_x^3\right)=1$$

The disagreement between quantum mechanics and the local realistic view becomes starkly apparent when the $\sigma_x^1 \sigma_x^2 \sigma_x^3$ measurement outcome is calculated.

$$\sigma_{xxx} := \text{kroncker}\left(\sigma_x, \text{kroncker}\left(\sigma_x, \sigma_x\right)\right) \quad \Psi^T \cdot \sigma_{xxx} \cdot \Psi = -1$$

Thus we see absolute disagreement between local realism and quantum mechanics. Local realism predicts an eigenvalue of +1 and quantum mechanics an eigenvalue of -1.

The validity of the reasoning above requires that $\sigma_x^1 \sigma_x^2 \sigma_x^3$ commutes with the other operators, which we now demonstrate.

$$\sigma_{xyy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{xyy} \rightarrow 0 \quad \sigma_{yxy} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yxy} \rightarrow 0 \quad \sigma_{yyx} \cdot \sigma_{xxx} - \sigma_{xxx} \cdot \sigma_{yyx} \rightarrow 0$$

Appendix

The tensor product of three vectors is shown below.

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} \otimes \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} ce \\ cf \\ de \\ df \end{pmatrix} = \begin{pmatrix} ace \\ acf \\ ade \\ adf \\ bce \\ bcf \\ bde \\ bdf \end{pmatrix}$$

Mathcad does not have a command for the vector tensor product, so it is necessary to develop a way of implementing it using *kroncker*, which requires square matrices. For this reason the spin vector is stored in the left column of a 2x2 matrix by augmenting the spin vector with the null vector. After all the matrix tensor products have been carried out using *kroncker* the final spin vector resides in the left column of the final square matrix. Next the *submatrix* command is used to save this column, discarding the rest of the matrix.

The Mathcad syntax for the tensor multiplication of three vectors is as follows.

$$\Psi(a, b, c) := \text{submatrix} \left[\text{kroncker} \left[\text{augment} \left[a, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \text{kroncker} \left[\text{augment} \left[b, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], \text{augment} \left[c, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \right], 1, 8, 1, 1 \right]$$

The initial spin state:

$$\frac{1}{\sqrt{2}} \cdot (\Psi(Sz_{\text{up}}, Sz_{\text{up}}, Sz_{\text{up}}) - \Psi(Sz_{\text{down}}, Sz_{\text{down}}, Sz_{\text{down}})) = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}$$

