Bohm, Bell and EPR

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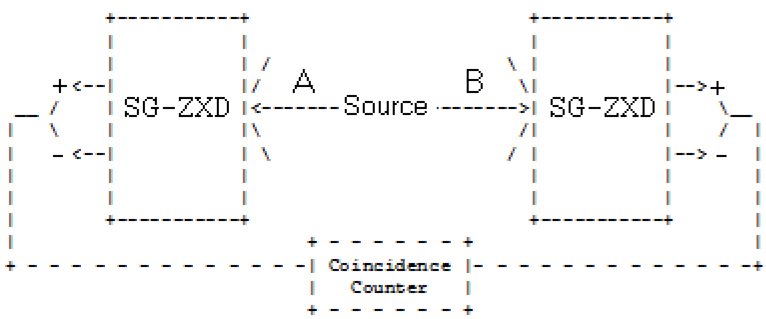
In 1951 David Bohm (*Quantum Theory*, pp. 614-623) proposed a gedanken experiment that further illuminated the conflict between local realism and quantum mechanics first articulated by Einstein, Podolsky and Rosen (EPR) in 1935. In his thought experiment a spin-1/2 pair is prepared in a singlet state and the individual particles travel in opposite directions on the y-axis to a pair of observers set up to measure spin in either the x- or z-direction. While Bohm's thought experiment clarified the conflict between quantum theory and classical realism, it did not provide for a direct experimental adjudication of the disagreement.

However, an extended version of Bohm's thought experiment suggested by John Bell in 1964 shows that there are experiments involving entangled spin systems for which a local hidden-variable theory makes predictions which are incompatible with those of quantum mechanics. For example, instead of measuring the spins in the x- and z-directions, use the x- or z-direction and another direction at some non-orthogonal angle, say 45 degrees, the diagonal or d-direction in the x-z plane. My goal in the following analysis is to bring the conflict between quantum theory and local realism into sharp focus as quickly as possible.

In this thought experiment a spin-1/2 pair is prepared in a singlet state and the individual particles travel in opposite directions on the y-axis to a pair of observers set up to measure spin in either the z-, x- or d-direction.



The spin eigenfunctions in the x-z plane:

Spin-up Eigenvalue +1 $\quad S_u(\theta) := \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$

Spin-down Eigenvalue -1 $\quad S_d(\theta) := \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$

The entangled singlet state expressed in several ways:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}_A \otimes \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}_B - \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}_A \otimes \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}_B\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right] = \frac{1}{\sqrt{2}}\left[|\rightarrow\rangle_A|\leftarrow\rangle_B - |\leftarrow\rangle_A|\rightarrow\rangle_B\right] = \frac{1}{\sqrt{2}}\left[|\nearrow\rangle_A|\swarrow\rangle_B - |\swarrow\rangle_A|\nearrow\rangle_B\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Spin operators in units of h/4π:

$$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_d := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Quantum mechanical expectation values for the various measurements observers A and B can make.

$$\Psi^T \cdot \text{kronecker}(\sigma_z, \sigma_z) \cdot \Psi = -1 \qquad \Psi^T \cdot \text{kronecker}(\sigma_x, \sigma_x) \cdot \Psi = -1 \qquad \Psi^T \cdot \text{kronecker}(\sigma_d, \sigma_d) \cdot \Psi = -1$$

$$\Psi^T \cdot \text{kronecker}(\sigma_z, \sigma_x) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(\sigma_z, \sigma_d) \cdot \Psi = -0.707 \qquad \Psi^T \cdot \text{kronecker}(\sigma_x, \sigma_d) \cdot \Psi = -0.707$$

The spin operators in the x-, d- and z-directions do not commute.

$$\sigma_z \cdot \sigma_x - \sigma_x \cdot \sigma_z = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \qquad \sigma_z \cdot \sigma_d - \sigma_d \cdot \sigma_z = \begin{pmatrix} 0 & 1.414 \\ -1.414 & 0 \end{pmatrix} \qquad \sigma_x \cdot \sigma_d - \sigma_d \cdot \sigma_x = \begin{pmatrix} 0 & -1.414 \\ 1.414 & 0 \end{pmatrix}$$

The following table presents a hidden-variable (HV) model for the singlet spin state and the expectation values for the various measurements observers A and B can make.

Spin A	Spin B	$\hat{S}_z(a) \cdot \hat{S}_z(b)$	$\hat{S}_x(a) \cdot \hat{S}_x(b)$	$\hat{S}_d(a) \cdot \hat{S}_d(b)$	$\hat{S}_z(a) \cdot \hat{S}_x(b)$	$\hat{S}_z(a) \cdot \hat{S}_d(b)$	$\hat{S}_x(a) \cdot \hat{S}_d(b)$
$\lvert\uparrow\rangle\lvert\rightarrow\rangle\lvert\nearrow\rangle$	$\lvert\downarrow\rangle\lvert\leftarrow\rangle\lvert\swarrow\rangle$	−1	−1	−1	−1	−1	−1
$\lvert\uparrow\rangle\lvert\rightarrow\rangle\lvert\swarrow\rangle$	$\lvert\downarrow\rangle\lvert\leftarrow\rangle\lvert\nearrow\rangle$	−1	−1	−1	−1	1	1
$\lvert\uparrow\rangle\lvert\leftarrow\rangle\lvert\nearrow\rangle$	$\lvert\downarrow\rangle\lvert\rightarrow\rangle\lvert\swarrow\rangle$	−1	−1	−1	1	−1	1
$\lvert\downarrow\rangle\lvert\rightarrow\rangle\lvert\nearrow\rangle$	$\lvert\uparrow\rangle\lvert\leftarrow\rangle\lvert\swarrow\rangle$	−1	−1	−1	1	1	−1
Expectation	Value	−1	−1	−1	0	0	0
Quantum	Value	−1	−1	−1	0	−0.707	−0.707

We see that the hidden-variable model agrees with quantum mechanics for the z-z, x-x, d-d and z-x measurement choices, but disagrees with z-d and x-d experiments. Furthermore, quantum mechanics says that the hidden-variable spin states in the first two columns are not valid because the spin operators in the z-, x-, and d-directions do not commute. According to quantum mechanics the spins in these directions cannot have simultaneous eigenstates because, like position and momentum, they are conjugate or incompatible observables.

The perfect anti-correlations for the z-z, x-x and d-d experiments are a direct and obvious consequence of the fact that the spins occupy a singlet state. The z-x, z-d and x-d experiments require more detailed analyses. We begin with z-x observations, and then use the same approach for the z-d and x-d cases.

If A has spin up in the z-direction, B has spin down in the z-direction.

The probability B has spin up in the x-drection giving a composite eigenvalue of +1 is $\left(\left|S_u(x)^T \cdot S_d(z)\right|\right)^2 = 0.5$

The probability B has spin down in the x-drection giving a composite eigenvalue of -1 is $\left(\left|S_d(x)^T \cdot S_d(z)\right|\right)^2 = 0.5$

If A has spin down in the z-direction, B has spin up in the z-direction.

The probability B has spin down in the x-drection giving a composite eigenvalue of +1 is $\left(\left|S_d(x)^T \cdot S_u(z)\right|\right)^2 = 0.5$

The probability B has spin up in the x-drection giving a composite eigenvalue of -1 is $\left(\left|S_u(x)^T \cdot S_u(z)\right|\right)^2 = 0.5$

Because A will be found spin up half the time and spin down half the time, the z-x expectation value is,

$$\frac{1}{2}\cdot\left[\left(\left|S_u(x)^T\cdot S_d(z)\right|\right)^2-\left(\left|S_d(x)^T\cdot S_d(z)\right|\right)^2\right]+\frac{1}{2}\cdot\left[\left(\left|S_d(x)^T\cdot S_u(z)\right|\right)^2-\left(\left|S_u(x)^T\cdot S_u(z)\right|\right)^2\right]=0$$

If A has spin up in the z-direction, B has spin down in the z-direction.

The probability B has spin up in the d-drection giving a composite eigenvalue of +1 is $\left(\left|S_u(d)^T\cdot S_d(z)\right|\right)^2=0.146$

The probability B has spin down in the d-drection giving a composite eigenvalue of -1 is $\left(\left|S_d(d)^T\cdot S_d(z)\right|\right)^2=0.854$

If A has spin down in the z-direction, B has spin up in the z-direction.

The probability B has spin down in the d-drection giving a composite eigenvalue of +1 is $\left(\left|S_d(d)^T\cdot S_u(z)\right|\right)^2=0.146$

The probability B has spin up in the d-drection giving a composite eigenvalue of -1 is $\left(\left|S_u(d)^T\cdot S_u(z)\right|\right)^2=0.854$

Because A will be found spin up half the time and spin down half the time, the z-d expectation value is,

$$\frac{1}{2}\cdot\left[\left(\left|S_u(d)^T\cdot S_d(z)\right|\right)^2-\left(\left|S_d(d)^T\cdot S_d(z)\right|\right)^2\right]+\frac{1}{2}\cdot\left[\left(\left|S_d(d)^T\cdot S_u(z)\right|\right)^2-\left(\left|S_u(d)^T\cdot S_u(z)\right|\right)^2\right]=-0.707$$

If A has spin up in the x-direction, B has spin down in the x-direction.

The probability B has spin up in the d-drection giving a composite eigenvalue of +1 is $\left(\left|S_u(d)^T\cdot S_d(x)\right|\right)^2=0.146$

The probability B has spin down in the d-drection giving a composite eigenvalue of -1 is $\left(\left|S_d(d)^T\cdot S_d(x)\right|\right)^2=0.854$

If A has spin down in the x-direction, B has spin up in the x-direction.

The probability B has spin down in the d-drection giving a composite eigenvalue of +1 is $\left(\left|S_d(d)^T\cdot S_u(x)\right|\right)^2=0.146$

The probability B has spin up in the d-drection giving a composite eigenvalue of -1 is $\left(\left|S_u(d)^T\cdot S_u(x)\right|\right)^2=0.854$

Because A will be found spin up half the time and spin down half the time, the x-d expectation value is

$$\frac{1}{2}\cdot\left[\left(\left|S_u(d)^T\cdot S_d(x)\right|\right)^2-\left(\left|S_d(d)^T\cdot S_d(x)\right|\right)^2\right]+\frac{1}{2}\cdot\left[\left(\left|S_d(d)^T\cdot S_u(x)\right|\right)^2-\left(\left|S_u(d)^T\cdot S_u(x)\right|\right)^2\right]=-0.707$$

"We conclude then that no theory of mechanically determined hidden variables can lead to *all* of the results of the quantum theory." David Bohm, *Quantum Theory*, Dover Publications, Inc., 1951, page 623.

Global definitions of the x, d and z angles: $x\equiv\frac{\pi}{2}$ $\quad d\equiv\frac{\pi}{4}$ $\quad z\equiv 0$