An Illustration of the Deutsch-Jozsa Algorithm

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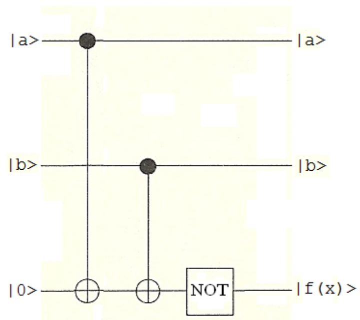
The following circuit, U_f, produces the table of results to its right. The top wires carry the value of x and the circuit places f(x) on the bottom wire. As is shown in the previous tutorial this circuit can also operate in parallel accepting as input all x-values and returning on the bottom wire a superposition of all values of f(x).

U_f =

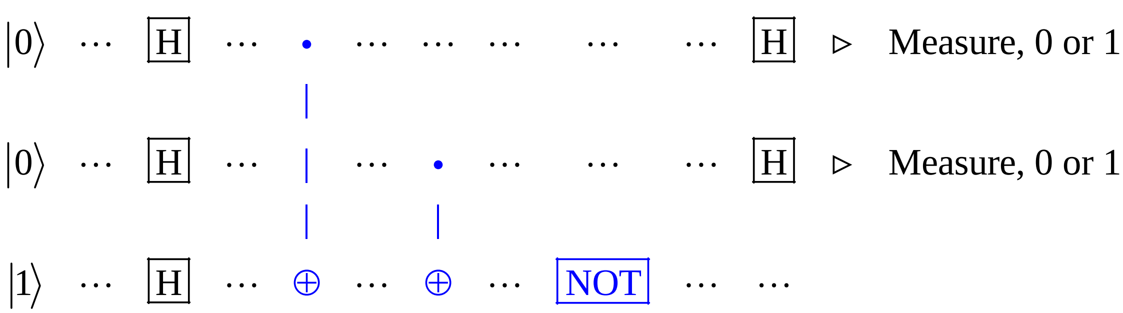


$$\begin{pmatrix} x & 0 & 1 & 2 & 3 \\ f(x) & 1 & 0 & 0 & 1 \end{pmatrix}$$

where

|0> = |0>|0>
|1> = |0>|1>
|2> = |1>|0>
|3> = |1>|1>

The function belongs to the balanced category because it produces 0 and 1 with equal frequency. The modification of this circuit (Deutsch-Jozsa algoritm) highlighted below answers the question of whether the function is constant or balanced (see Julian Brown, *The Quest for the Quantum Computer*, page 298). Naturally we already know the answer, so this is a simple demonstration that the circuit works.



The input is |0>|0>|1>: $\Psi_{in} := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$

The following matrices are required to execute the circuit. $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $NOT := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad CnNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The quantum circuit is assembled out of these matrices using tensor (kronecker) multiplication.

$$U_f := \text{kronecker}(I, \text{kronecker}(I, NOT)) \cdot \text{kronecker}(I, CNOT) \cdot CnNOT$$

$$\text{QuantumCircuit} := \text{kronecker}(H, \text{kronecker}(H, I)) \cdot U_f \cdot \text{kronecker}(H, \text{kronecker}(H, H))$$

Operation of the quantum circuit on the input vector yields the following result which is written as a product of three qubits on the right.

$$\text{QuantumCircuit} \cdot \Psi_{in} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \\ 0.707 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

According to the Deutsch-Jozsa scheme, if both wires are |0> the function is constant, but if at least one wire is |1> the function is balanced. We see by inspection that both wires are |1> indicating that the function is balanced.

The measurements on the top wires can be simulated with projection operators |0><0| and |1><1|, and confirm that the function is not constant but belongs to the balanced category.

The first qubit is not |0>.
$$\left[\text{kronecker}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T, \text{kronecker}(I, I)\right] \cdot \text{QuantumCircuit} \cdot \Psi_{in}\right]^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The second qubit is not |0>.
$$\left[\text{kronecker}\left[I, \text{kronecker}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T, I\right]\right] \cdot \text{QuantumCircuit} \cdot \Psi_{in}\right]^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The first qubit is |1>.
$$\left[\text{kronecker}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T, \text{kronecker}(I, I)\right] \cdot \text{QuantumCircuit} \cdot \Psi_{in}\right]^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 0.707 \end{pmatrix}$$

The second qubit is |1>.
$$\left[\text{kronecker}\left[I, \text{kronecker}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T, I\right]\right] \cdot \text{QuantumCircuit} \cdot \Psi_{in}\right]^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 0.707 \end{pmatrix}$$