The Discrete or Quantum Fourier Transform

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The continuous-variable Fourier transforms involving position and momentum are well known. In Dirac notation (see chapter 6 in *A Modern Approach to Quantum Mechanics* by John S. Townsend) they are,

$$\langle p|\Psi\rangle = \int \langle p|x\rangle\langle x|\Psi\rangle dx \qquad \text{and} \qquad \langle x|\Psi\rangle = \int \langle x|p\rangle\langle p|\Psi\rangle dp$$

where
$$\langle x|p\rangle = \langle p|x\rangle^* = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(i\frac{2\pi px}{h}\right) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(i\frac{px}{\hbar}\right)$$

Using the coordinate and momentum completeness relations

$$\int |x\rangle\langle x|dx = 1 \quad \text{and} \quad \int |p\rangle\langle p|dp = 1$$

we can write the following generic Fourier transforms.

$$\langle p| = \int \langle p|x\rangle\langle x|dx \quad \text{and} \quad \langle x| = \int \langle x|p\rangle\langle p|dp$$

By analogy a discrete Fourier transform between the k and j indices can be created.

$$\langle k| = \sum_{j=0}^{N-1} \langle k|j\rangle\langle j|$$

were, again, by analogy

$$\langle k|j\rangle = \frac{1}{\sqrt{N}}\exp\left(i\frac{2\pi}{N}k\cdot j\right)$$

so that

$$\langle k| = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\exp\left(i\frac{2\pi}{N}k\cdot j\right)\langle j|$$

Summing over the k index and projecting on to |Ψ> yields a system of linear equations.

$$\sum_{k=0}^{N-1}\langle k|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\exp\left(i\frac{2\pi}{N}k\cdot j\right)\langle j|\Psi\rangle$$

Like all such systems it is expressible in matrix form. For example, with N = 2 and $\begin{pmatrix}1\\0\end{pmatrix}$ as the operand we have,

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

Here the matrix operator is the well-known Hadamard transform. In this case it transforms spin-up in the z-direction to spin-up in the x-direction, or horizontal polarization to diagonal polarization, etc. Naturally it transforms spin-up in the x-direction to spin-up in the z-direction.

$$\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

This, of course, also occurs with the continuous-variable Fourier transfroms.

$$\langle x|\Psi\rangle \xrightarrow{FT} \langle p|\Psi\rangle \xrightarrow{FT} \langle x|\Psi\rangle$$

The Mathcad implimentation of the descrete or quantum Fourier transform (QFT) is now demonstrated.

$N := 2 \qquad m := 0 .. N-1 \qquad n := 0 .. N-1 \qquad QFT_{m,n} := \frac{1}{\sqrt{N}} \cdot \exp\left(i \cdot \frac{2 \cdot \pi \cdot m \cdot n}{N}\right)$

$$QFT = \begin{pmatrix}0.707 & 0.707\\0.707 & -0.707\end{pmatrix}$$

$QFT \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0.707\\0.707\end{pmatrix} \qquad QFT \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$QFT \cdot \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0.707\\-0.707\end{pmatrix} \qquad QFT \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

These calculation demonstrate that the QFT is a unitary operator: $QFT \cdot QFT = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$