

# Quantum Error Correction

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This tutorial deals with quantum error correction as presented by Julian Brown on pages 274-278 in *The Quest for the Quantum Computer*. Brown's three-qubit example includes an input qubit and two ancillary qubits in the initial state  $|\Psi_{00}\rangle$ . This state is encoded and subsequently decoded, with the possibility that in between it acquires an error. The quantum circuit demonstrates how correction occurs if a qubit is flipped due to decoherence at the intermediate state.

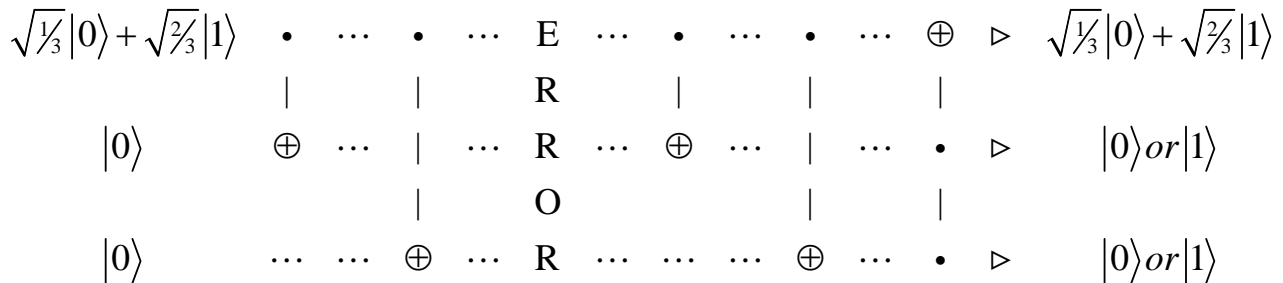
The quantum gates required for the error correction algorithm are:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{CnNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{IToffoli} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The encoding and decoding elements of the circuit in terms of these gates are shown in the appropriate place above the circuit diagram.

Encode := CnNOT.kronecker(CNOT, I)

Decode := IToffoli.Encode



Given an initial state, the encoding step creates an entangled state as demonstrated below.

Initial State

Encode

Encoded state

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{Encode} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.577 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.816 \end{pmatrix} \quad \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

As an initial example, it is assumed that between encoding and decoding no errors are introduced to the encoded state. This case demonstrates that decoding simply returns the initial state. Subsequent to this the operation of the circuit when errors occur on each of the wires are examined. These cases demonstrate that the original state appears on the top wire at the completion of the error correction circuit.

No errors:

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$\text{Decode} \cdot \begin{pmatrix} 0.577 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.816 \end{pmatrix} = \begin{pmatrix} 0.577 \\ 0 \\ 0 \\ 0 \\ 0.816 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Next it is shown that if the input state,  $|\Psi\rangle$ , is corrupted that the decoder corrects the error and returns the original  $|\Psi\rangle$  to the top wire of the circuit. In the example shown below the top qubit is flipped.

Top qubit flipped:

$$\begin{pmatrix} 0 \\ \sqrt{\frac{1}{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Decode} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.577 \\ 0 \\ 0 \\ 0 \\ 0.816 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

The circuit can also be expressed using Dirac notation. Truth tables for the gates are provided in the Appendix.

$$\begin{aligned} & \left( \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle \right) |00\rangle \xrightarrow{\text{encode}} \sqrt{\frac{1}{3}}|000\rangle + \sqrt{\frac{2}{3}}|111\rangle \xrightarrow{\text{flip top qubit}} \sqrt{\frac{1}{3}}|100\rangle + \sqrt{\frac{2}{3}}|011\rangle \\ & \xrightarrow{\text{CNOT, I}} \sqrt{\frac{1}{3}}|110\rangle + \sqrt{\frac{2}{3}}|011\rangle \xrightarrow{\text{CnNOT}} \sqrt{\frac{1}{3}}|111\rangle + \sqrt{\frac{2}{3}}|011\rangle \\ & \xrightarrow{\text{InToffoli}} \sqrt{\frac{1}{3}}|011\rangle + \sqrt{\frac{2}{3}}|111\rangle = \left( \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle \right) |11\rangle \end{aligned}$$

Naturally the ancillary qubits are also susceptible to errors. The following examples show that if a qubit flip occurs on the middle or bottom wire, the circuit still functions properly.

Middle qubit flipped:

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Decode:

$$\begin{pmatrix} 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.577 \\ 0 \\ 0 \\ 0.816 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

$$\begin{aligned} & (\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle)|00\rangle \xrightarrow{\text{encode}} \sqrt{\frac{1}{3}}|000\rangle + \sqrt{\frac{2}{3}}|111\rangle \xrightarrow[\text{qubit}]{\text{flip middle}} \sqrt{\frac{1}{3}}|010\rangle + \sqrt{\frac{2}{3}}|101\rangle \\ & \xrightarrow{\text{CNOT, I}} \sqrt{\frac{1}{3}}|010\rangle + \sqrt{\frac{2}{3}}|111\rangle \xrightarrow{\text{CnNOT}} \sqrt{\frac{1}{3}}|010\rangle + \sqrt{\frac{2}{3}}|110\rangle \\ & \xrightarrow{\text{InToffoli}} \sqrt{\frac{1}{3}}|010\rangle + \sqrt{\frac{2}{3}}|110\rangle = (\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle)|10\rangle \end{aligned}$$

Bottom qubit flipped:

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

Decode:

$$\begin{pmatrix} 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.577 \\ 0 \\ 0 \\ 0.816 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

$$\begin{aligned} & (\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle)|00\rangle \xrightarrow{\text{encode}} \sqrt{\frac{1}{3}}|000\rangle + \sqrt{\frac{2}{3}}|111\rangle \xrightarrow[\text{qubit}]{\text{flip bottom}} \sqrt{\frac{1}{3}}|001\rangle + \sqrt{\frac{2}{3}}|110\rangle \\ & \xrightarrow{\text{CNOT, I}} \sqrt{\frac{1}{3}}|001\rangle + \sqrt{\frac{2}{3}}|100\rangle \xrightarrow{\text{CnNOT}} \sqrt{\frac{1}{3}}|001\rangle + \sqrt{\frac{2}{3}}|101\rangle \\ & \xrightarrow{\text{InToffoli}} \sqrt{\frac{1}{3}}|001\rangle + \sqrt{\frac{2}{3}}|101\rangle = (\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle)|01\rangle \end{aligned}$$

## Appendix

### CNOT

Decimal	Binary	to	Binary	Decimal
0	00	to	00	0
1	01	to	01	1
2	10	to	11	3
3	11	to	10	2

### CnNOT

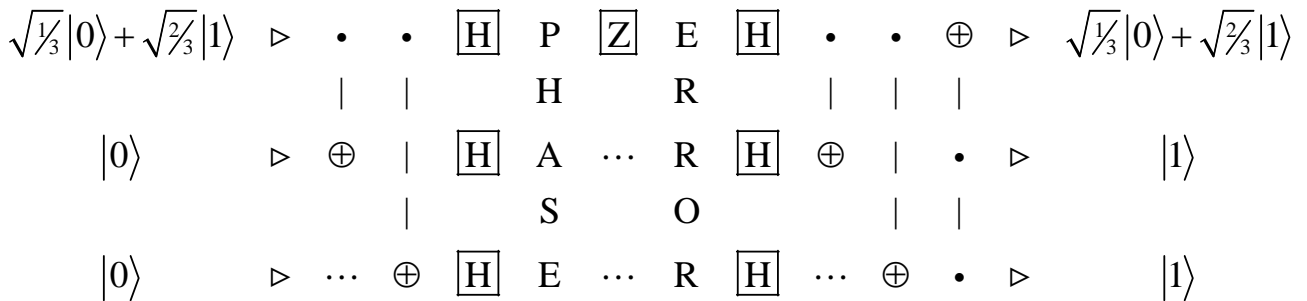
Decimal	Binary	to	Binary	Decimal
0	000	to	000	0
1	001	to	001	1
2	010	to	010	2
3	011	to	011	3
4	100	to	101	5
5	101	to	100	4
6	110	to	111	7
7	111	to	110	6

### IToffoli

Decimal	Binary	to	Binary	Decimal
0	000	to	000	0
1	001	to	001	1
2	010	to	010	2
3	011	to	111	7
4	100	to	100	4
5	101	to	101	5
6	110	to	110	6
7	111	to	011	3

The modifications to the bit-flip circuit shown below convert it to a circuit which corrects a phase error introduced on the top wire. The H and Z gates are provided below.

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$\text{Encode}' := \text{kroncker}(H, \text{kroncker}(H, H)) \cdot \text{Encode} \quad \text{Decode}' := \text{Decode} \cdot \text{kroncker}(H, \text{kroncker}(H, H))$$

$$\text{PhaseErrorCircuit} := \text{Decode}' \cdot \text{kroncker}(Z, \text{kroncker}(I, I)) \cdot \text{Encode}'$$

### Initial State

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

### Circuit Operation

$$\text{PhaseErrorCircuit} \cdot \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.577 \\ 0 \\ 0 \\ 0 \\ 0.816 \end{pmatrix}$$

### Final State

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$