A Thought Experiment Reveals the Conflict Between Quantum Theory and Local Realism

Frank Rioux

In order to explore the conflict between quantum mechanics and local realism a spin-1/2 pair is prepared in a singlet state and the individual particles travel in opposite directions on the y-axis to a pair of observers set up to measure spin with Stern-Gerlach magnets oriented in either the d- or z-direction, where d (diagonal) refers to a 45 degree rotation clockwise from the vertical z-axis. The entangled singlet spin state can be written using either the z- or d-direction spin eigenstates.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right] = \frac{1}{\sqrt{2}}\left[|\nearrow\rangle_1|\swarrow\rangle_2 - |\swarrow\rangle_1|\nearrow\rangle_2\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}^T$$

The spin-up and spin-down eigenstates in the x-z plane are shown below in general and explicity for the z- and d-directions.

Spin-up
Eigenvalue +1
$$\varphi_u(\varphi) := \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right)\\ \sin\left(\frac{\varphi}{2}\right)\end{pmatrix}$$

Spin-down
Eigenvalue -1
$$\varphi_d(\varphi) := \begin{pmatrix}-\sin\left(\frac{\varphi}{2}\right)\\ \cos\left(\frac{\varphi}{2}\right)\end{pmatrix}$$

$$z := 0\cdot\text{deg} \qquad d := 45\cdot\text{deg} \qquad \varphi_u(z) = \begin{pmatrix}1\\0\end{pmatrix} \qquad \varphi_d(z) = \begin{pmatrix}0\\1\end{pmatrix} \qquad \varphi_u(d) = \begin{pmatrix}0.924\\0.383\end{pmatrix} \qquad \varphi_d(d) = \begin{pmatrix}-0.383\\0.924\end{pmatrix}$$

If the observers measure their spins in the same direction quantum mechanics predicts they will get opposite values due to the singlet nature of the spin state. In other words, the combined expectation value is -1 for these measurements.

A realist believes that objects have well-defined properties prior to and independent of observation. The first four columns of the following table provide a local realist's explanation of this result. Specific z- and d-spin states are assigned to the particles in the first two columns, with each particle in one of four equally probable spin orientations consistent with the composite singlet state. The next two columns show that these assignments agree with the quantum prediction. Unfortunately quantum mechanics does not accept the legitimacy of the local realist's spin states because S_z and S_d are noncommuting spin operators and therefore cannot have simultaneous eigenvalues.

Particle 1	Particle 2	$\hat{S}_z(1)\cdot\hat{S}_z(2)$	$\hat{S}_d(1)\cdot\hat{S}_d(2)$	$\hat{S}_z(1)\cdot\hat{S}_d(2)$	$\hat{S}_d(1)\cdot\hat{S}_z(2)$
$\|\uparrow\rangle\|\nearrow\rangle$	$\|\downarrow\rangle\|\swarrow\rangle$	−1	−1	−1	−1
$\|\uparrow\rangle\|\swarrow\rangle$	$\|\downarrow\rangle\|\nearrow\rangle$	−1	−1	1	1
$\|\downarrow\rangle\|\nearrow\rangle$	$\|\uparrow\rangle\|\swarrow\rangle$	−1	−1	1	1
$\|\downarrow\rangle\|\swarrow\rangle$	$\|\uparrow\rangle\|\nearrow\rangle$	−1	−1	−1	−1
Realist	Value	−1	−1	0	0
Quantum	Value	−1	−1	−0.707	−0.707

Putting this issue asside momentarily, if one spin is measured in the z-direction and the other in the d-direction, a realist predicts an expectation value of zero as shown in the last two columns of the table. However, quantum theory predicts an expectation value of -0.707. If one of the particles is observed to be spin-up in the z-direction (eigenvalue +1), the other is spin-down in the z-direction. The probability it will be found on measurement to be spin-up in the d-direction yielding a composite eigenvalue of +1 is 0.146. The probability it will be found to be spin-down in the d-direction yielding a composite eigenvalue of -1 is 0.854.

$$\left(\left|\varphi_u(d)^T\cdot\varphi_d(z)\right|\right)^2 = 0.146 \qquad \left(\left|\varphi_d(d)^T\cdot\varphi_d(z)\right|\right)^2 = 0.854 \qquad \left(\left|\varphi_u(d)^T\cdot\varphi_d(z)\right|\right)^2 - \left(\left|\varphi_d(d)^T\cdot\varphi_d(z)\right|\right)^2 = -0.707$$

This brief analysis demonstrates that there are conceptually simple, Stern-Gerlach like, experiments on spin-1/2 systems which can adjudicate the conflict between local realism and quantum mechanics.

A Quantum Simulation

This thought experiment is simulated using the following quantum circuit. As shown below the results are in agreement with the previous theoretical quantum calculations. The initial Hadamard and CNOT gates create the singlet state from the |11> input. $R_z(\theta)$ rotates spin B. The final Hadamard gates prepare the system for measurement. See arXiv:1712.05642v2 for further detail.

$$\begin{array}{lllllllll} \text{Spin A} & |1\rangle & \triangleright & H & \bullet & \cdots & H & \triangleright & \text{Measure 0 or 1: Eigenvalue } +1 \text{ or -1} \\ & & & & | & & & & \\ \text{Spin B} & |1\rangle & \triangleright & \cdots & \oplus & R_z(\theta) & H & \triangleright & \text{Measure 0 or 1: Eigenvalue } +1 \text{ or -1} \end{array}$$

The quantum gates required to execute this circuit:

Identity | Hadamard gate | R_z rotation | Controlled NOT

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad R_z(\theta) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i \cdot \theta} \end{pmatrix} \qquad CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The operator representing the circuit is constructed from the matrix operators provided above.

$$Op(\theta) := \text{kronecker}(H,H) \cdot \text{kronecker}\left(I, R_z(\theta)\right) CNOT\ \text{kronecker}(H,I)$$

There are four equally likely measurement outcomes with the eigenvalues and overall expectation values shown below.

|00> eigenvalue +1 $\quad \left[\left|\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \cdot Op(0 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0$

|01> eigenvalue -1 $\quad \left[\left|\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \cdot Op(0 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0.5$

|10> eigenvalue -1 $\quad \left[\left|\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}^T \cdot Op(0 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0.5$

|11> eigenvalue +1 $\quad \left[\left|\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T \cdot Op(0 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0$

Epectation value: $0 - 0.5 - 0.5 + 0 = -1$

|00> eigenvalue +1 $\quad \left[\left|\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \cdot Op(45 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0.073$

|01> eigenvalue -1 $\quad \left[\left|\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T \cdot Op(45 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0.427$

|10> eigenvalue -1 $\quad \left[\left|\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}^T \cdot Op(45 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0.427$

|11> eigenvalue +1 $\quad \left[\left|\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T \cdot Op(45 \cdot deg) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right|\right]^2 = 0.073$

Epectation value: $0.073 - 0.427 - 0.427 + 0.073 = -0.708$