A Surgical Adjudication of the Conflict Between Quantum Theory and Local Realism

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In 1951 David Bohm (*Quantum Theory*, pp. 614-623) proposed a gedanken experiment that further illuminated the conflict between local realism and quantum mechanics first articulated by Einstein, Podolsky and Rosen (EPR) in 1935. In his thought experiment a spin-1/2 pair is prepared in a singlet state and the individual particles travel in opposite directions on the y-axis to a pair of observers set up to measure spin in either the x- or z-direction. While Bohm's thought experiment clarified the conflict between quantum theory and classical realism, it did not provide for a direct experimental adjudication of the disagreement.

However, a slightly modified version of Bohm's thought experiment shows that there are experiments involving entangled spin systems for which a local hidden-variable theory makes predictions which are incompatible with those of quantum mechanics. For example, instead of measuring the spins in the x- and z-directions, use the z-direction and another direction at some non-orthogonal angle to the z-axis, say 45 degrees, the diagonal or d-direction. My goal in the following analysis is to bring the conflict between quantum theory and local realism into sharp focus as quickly as possible.

The z-direction spin operator and its eigenvalues and eigenvectors:

$$S_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvals}(S_z) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{eigenvecs}(S_z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The singlet spin state written using the z-direction eigenstates:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

The d-direction spin operator and its eigenvalues and eigenvectors:

$$S_d := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{eigenvals}(S_d) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{eigenvecs}(S_d) = \begin{pmatrix} 0.924 & -0.383 \\ 0.383 & 0.924 \end{pmatrix}$$

The singlet spin state written using the d-direction eigenstates:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\nearrow\rangle_1|\swarrow\rangle_2 - |\swarrow\rangle_1|\nearrow\rangle_2\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \otimes \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} - \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \otimes \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

The calculations below show that the individual spin measurements are totally random yielding expectation values of zero in both the z- and d-directions. The identity operator, do nothing, is required for these calculations. However, the expectation value observed when both observers jointly measure the same spin direction is -1, because in the singlet state the spins have opposite orientations in both spin bases.

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Psi^T \cdot \text{kronecker}(S_z, I) \cdot \Psi = 0$ $\quad \Psi^T \cdot \text{kronecker}(I, S_z) \cdot \Psi = 0$ $\quad \Psi^T \cdot \text{kronecker}(S_z, S_z) \cdot \Psi = -1$

$\Psi^T \cdot \text{kronecker}(S_d, I) \cdot \Psi = 0$ $\quad \Psi^T \cdot \text{kronecker}(I, S_d) \cdot \Psi = 0$ $\quad \Psi^T \cdot \text{kronecker}(S_d, S_d) \cdot \Psi = -1$

The first four columns of the following table provide a local realist's explanation of these calculations. Specific z- and d-spin states are assigned to the particles in the first two columns, with each particle in one of four equally probable spin orientations consistent with the composite singlet state. The next two columns show that these assignments agree with the quantum calculations.

Particle 1	Particle 2	$\hat{S}_z(1)\cdot\hat{S}_z(2)$	$\hat{S}_d(1)\cdot\hat{S}_d(2)$	$\hat{S}_z(1)\cdot\hat{S}_d(2)$	$\hat{S}_d(1)\cdot\hat{S}_z(2)$
$\lvert\uparrow\rangle\lvert\nearrow\rangle$	$\lvert\downarrow\rangle\lvert\swarrow\rangle$	-1	-1	-1	-1
$\lvert\uparrow\rangle\lvert\swarrow\rangle$	$\lvert\downarrow\rangle\lvert\nearrow\rangle$	-1	-1	1	1
$\lvert\downarrow\rangle\lvert\nearrow\rangle$	$\lvert\uparrow\rangle\lvert\swarrow\rangle$	-1	-1	1	1
$\lvert\downarrow\rangle\lvert\swarrow\rangle$	$\lvert\uparrow\rangle\lvert\nearrow\rangle$	-1	-1	-1	-1
Expectation	Value	-1	-1	0	0

At this point one may ask, "Where's the problem? The quantum and classical pictures are in agreement on the prediction of experimental results." The difficulty is that quantum mechanics does not accept the legitimacy of the states shown in the table on the left. One way to state the problem is to note that S_d and S_z are non-commuting operators. This means that spin in the d- and z-directions cannot simultaneously have well-defined values because like position and momentum they are conjugate observables.

The S_z-S_d commutator: $\quad S_z \cdot S_d - S_d \cdot S_z = \begin{pmatrix} 0 & 1.414 \\ -1.414 & 0 \end{pmatrix}$

In other words, the z-eigenstates are superpositions of d-eigenstates and vice versa.

$$.924\cdot\begin{pmatrix} .924 \\ .383 \end{pmatrix} - .383\cdot\begin{pmatrix} -.383 \\ .924 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad .383\cdot\begin{pmatrix} .924 \\ .383 \end{pmatrix} + .924\cdot\begin{pmatrix} -.383 \\ .924 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$.924\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + .383\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \qquad .924\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} - .383\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix}$$

Of course, the realist says this argument simply reveals that quantum mechanics does not provide a complete representation of reality because it is unable to assign definite values to all observables prior to and independent of measurement.

Fortunately another set of measurements can settle the dispute. If one spin is measured in the z-direction and the other in the d-direction, local realism predicts an expectation value of zero as shown in the last two columns of the table above. Quantum theory, however, predicts -0.707.

$\Psi^T \cdot \text{kronecker}(S_z, S_d) \cdot \Psi = -0.707$ $\qquad \Psi^T \cdot \text{kronecker}(S_d, S_z) \cdot \Psi = -0.707$

This brief analysis demonstrates that there are conceptually simple, Stern-Gerlach like, experiments on spin-1/2 systems for which the predictions of quantum mechanics and local realism are in disagreement.