An Extension of Bohm's EPR Experiment

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> Quantum theory is both stupendously successful as an account of the small-scale structure of the world and it is also the subject of an unresolved debate and dispute about its interpretation. J. C. Polkinghorne, *The Quantum World*, p. 1.

In 1951 David Bohm proposed a gedanken experiment that further illuminated the conflict between local realism and quantum mechanics first articulated by Einstein, Podolsky and Rosen (EPR) in 1935. In this thought experiment a spin-1/2 pair is prepared in a singlet state and the individual particles travel in opposite directions on the y-axis to a pair of observers set up to measure spin in either the x- or z-direction.

In this summary tensor algebra will be used to analyze Bohm's thought experiment. The vector states and matrix operators required are provided below. The Appendix reviews basic vector/matrix operations.

Spin eigenvectors: $S_{zu} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad S_{zd} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad S_{xu} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad S_{xd} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Spin operators in units of h/4π: $S_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad S_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad$ Identity: $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

According to established quantum principles the singlet state for fermions is an entangled superposition written as follows in tensor format in the z-direction spin basis.

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|S_{zu}\right\rangle_1 \left|S_{zd}\right\rangle_2 - \left|S_{zd}\right\rangle_1 \left|S_{zu}\right\rangle_2\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

First we calculate the expectation values for spin measurements in the z- and x-directions using the eigenvectors of the respective spin operators. We see that spin-up yields +1 and spin-down -1.

$$S_{zu}^T \cdot S_z \cdot S_{zu} = 1 \qquad S_{zd}^T \cdot S_z \cdot S_{zd} = -1 \qquad S_{xu}^T \cdot S_x \cdot S_{xu} = 1 \qquad S_{xd}^T \cdot S_x \cdot S_{xd} = -1$$

Consequently the expectation value observed when both observers jointly measure the same spin direction is -1, because in the singlet state the spins have opposite orientations (**kronecker** performs the tensor multiplication of two matrices, as illustrated in the Appendix).

$$\Psi^T \cdot \text{kronecker}(S_z, S_z) \cdot \Psi = -1 \qquad \Psi^T \cdot \text{kronecker}(S_x, S_x) \cdot \Psi = -1$$

Note that while the entangled singlet spin state shown above is written in the z-basis, it is the same to an over-all phase using the x-direction eigenvectors.

$$\Psi = \frac{1}{\sqrt{2}}\left[\left|S_{xd}\right\rangle_1 \left|S_{xu}\right\rangle_2 - \left|S_{xu}\right\rangle_1 \left|S_{xd}\right\rangle_2\right] = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Next we show that in spite of the strong correlation shown above, individual spin measurements are totally random yielding expectation values of zero in both the z- and x-directions.

$$\Psi^T \cdot \text{kronecker}(S_z, I) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(I, S_z) \cdot \Psi = 0$$

$$\Psi^T \cdot \text{kronecker}(S_x, I) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(I, S_x) \cdot \Psi = 0$$

Quantum mechanics predicts the following results when the observers make different spin measurements on the particles.

$$\Psi^T \cdot \text{kronecker}(S_z, S_x) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(S_x, S_z) \cdot \Psi = 0$$

From a classical realist position the results for all the previous quantum calculations can be explained by assigning specific x- and z-spin states to the particles, as shown in the table below on the left (see Townsend's *A Modern Approach to Quantum Mechanics*, page 135). Each particle can be in any one of four equally probable spin states, and taken together the particles form four equally probable joint spin states. In other words, the particles are in well-defined, although unknown, spin states prior to measurement. Measurement, according to a realist, simply reveals a pre-existing state. The right-hand part of the table gives the joint measurement results expected given the spin states specified on the left. The bottom row gives the expectation (average) values under the assumptions stated above.

$$\begin{pmatrix} \text{Particle1} & ' & \text{Particle2} & ' & S_z(1)\cdot S_z(2) & S_x(1)\cdot S_x(2) & S_z(1)\cdot S_x(2) & S_x(1)\cdot S_z(2) \\ S_{zu}\cdot S_{xu} & ' & S_{zd}\cdot S_{xd} & ' & -1 & -1 & -1 & -1 \\ S_{zu}\cdot S_{xd} & ' & S_{zd}\cdot S_{xu} & ' & -1 & -1 & 1 & 1 \\ S_{zd}\cdot S_{xu} & ' & S_{zu}\cdot S_{xd} & ' & -1 & -1 & 1 & 1 \\ S_{zd}\cdot S_{xd} & ' & S_{zu}\cdot S_{xu} & ' & -1 & -1 & -1 & -1 \\ \text{Expectation} & ' & \text{Value} & ' & -1 & -1 & 0 & 0 \end{pmatrix}$$

At this point one may ask, "Where's the problem? The quantum and classical pictures are in agreement on the prediction of experimental results." The difficulty is that quantum mechanics does not accept the legitimacy of the states shown in the table on the left. One way to state the problem is to note that S_x and S_z are non-commuting operators. This means that according to quantum mechanics spin in the x- and z-directions cannot simultaneously have well-defined values (like position and momentum, they are conjugate observables).

The S_x-S_z commutator: $$S_z \cdot S_x - S_x \cdot S_z = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

For example, if measurement reveals that particle 1 has S_{zu} then particle 2 is definitely S_{zd}. But that means it cannot have a well-defined value for the x-direction spin. In fact S_{zd} is a superposition of S_{xu} and S_{xd}, as shown below, with $<S_x> = 0$.

$$S_{zd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (S_{xu} - S_{xd}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad S_{zd}^T \cdot S_x \cdot S_{zd} = 0$$

Or, suppose particle 1 is found to have S_{xd}, then particle 2 is S_{xu} which is a superposition of spin up and down in the z-direction and therefore $<S_z> = 0$.

$$S_{xu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (S_{zu} + S_{zd}) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad {S_{xu}}^T \cdot S_z \cdot S_{xu} = 0$$

To summarize, the states in the table are not valid in spite of their agreement with experimental results, because they give well-defined values to incompatible (according to quantum theory) observables. Of course, the realist says that these arguments simply indicate that quantum mechanics is not a complete theory because it cannot assign definite values to all elements of reality independent of measurement.

While Bohm's 1951 gedanken experiment clarified the conflict between quantum theory and classical realism, it did not provide for a direct experimental adjudication of the disagreement. That changed in 1964 with a theoretical analysis by John Bell that showed that there are experimental situations where the predictions of quantum mechanics and local realism are in disagreement. We look at one of them now.

A modified version of the thought experiment shows that there are experiments involving entangled spin systems for which a local hidden-variable theory gives predictions which are incompatible with those of quantum theory. Instead of measuring the spins in the z- and x-directions, measure one in the z-direction and the other at some non-orthogonal angle to the z-axis, say 45 degrees ($\pi/4$). The appropriate spin operator for this diagonal direction (an even superposition of S_x and S_z) and its eigenvalues and eigenvectors are given below, and the singlet spin state is written in the diagonal spin basis.

$$S_d := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{eigenvals}(S_d) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{eigenvecs}(S_d) = \begin{pmatrix} 0.924 & -0.383 \\ 0.383 & 0.924 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}\left[\left|S_{du}\right\rangle_A \left|S_{dd}\right\rangle_B - \left|S_{dd}\right\rangle_A \left|S_{du}\right\rangle_B \right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \otimes \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} - \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \otimes \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

As expected, perfect anti-correlation is observed if both spins are measured in the diagonal direction while the individual spin measurements are totally random with an expectation value of zero.

$$\Psi^T \cdot \text{kronecker}(S_d, S_d) \cdot \Psi = -1 \qquad \Psi^T \cdot \text{kronecker}(S_d, I) \cdot \Psi = 0 \qquad \Psi^T \cdot \text{kronecker}(I, S_d) \cdot \Psi = 0$$

Up to this point the calculations are consistent with those for the x- and z-directions, and would appear to justify the local realist in providing the following hidden-variable interpretation of the results.

Particle1	Particle2	$S_z(1) \cdot S_z(2)$	$S_d(1) \cdot S_d(2)$	$S_z(1) \cdot S_d(2)$	$S_d(1) \cdot S_z(2)$
$S_{zu} \cdot S_{du}$	$S_{zd} \cdot S_{dd}$	−1	−1	−1	−1
$S_{zu} \cdot S_{dd}$	$S_{zd} \cdot S_{du}$	−1	−1	1	1
$S_{zd} \cdot S_{du}$	$S_{zu} \cdot S_{dd}$	−1	−1	1	1
$S_{zd} \cdot S_{dd}$	$S_{zu} \cdot S_{du}$	−1	−1	−1	−1
Expectation	Value	−1	−1	0	0

However when the spins are measured in different directions (the z- and d-directions) quantum mechanics predicts that the expectation values are not zero as predicted by the hidden-variable model. According to quantum theory there is significant anti-correlation in the joint spin measurements.

$$\Psi^{T}\cdot \mathrm{kronecker}(S_z, S_d)\cdot \Psi = -0.707 \qquad \Psi^{T}\cdot \mathrm{kronecker}(S_d, S_z)\cdot \Psi = -0.707$$

On the basis of another thought experiment, Bohm wrote on page 623 of his 1951 treatise Quantum Theory, "We conclude then that no theory of mechanically determined hidden variables can lead to *all* of the results of the quantum theory."

The previous calculations can be simulated with the Quirk Quantum Simulator (algassert.com/quirk) using the following circuit. On the left |11> is the index for the creation of the singlet Bell state. A Bell state measurement is performed on the right and the |11> probability amplitude is the expectation value.

$\vert 1\rangle$	$\boxed{H}$	•	$\boxed{Z}$		$\boxed{Z}$		$\boxed{X}$		$\boxed{X}$		$\boxed{H}$		$\boxed{H}$		$\boxed{Z}$		$\boxed{Z}$		$\boxed{X}$	•	$\boxed{H}$	$\vert 0\rangle$ or $\vert 1\rangle$
		|		or		or		or		or		or		or		or		or		|		
$\vert 1\rangle$	⋯	⊕	$\boxed{Z}$		⋯		$\boxed{X}$		⋯		$\boxed{H}$		⋯		$\boxed{X}$		$\boxed{H}$		$\boxed{H}$	⊕	⋯	$\vert 0\rangle$ or $\vert 1\rangle$

Note that the Hadamard gate is identical to the diagonal spin operator: $\mathbf{H} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = S_d$

Appendix: Vector and Matrix Math

This addendum reviews basic vector and matrix operations. At the end it illustrates how vector and matrix tensor multiplication are implemented in Mathcad.

Vector inner product: $\begin{pmatrix}a & b\end{pmatrix}\cdot\begin{pmatrix}c\\ d\end{pmatrix} \rightarrow a\cdot c + b\cdot d \qquad \mathrm{tr}\left[\begin{pmatrix}a\\ b\end{pmatrix}\cdot\begin{pmatrix}c & d\end{pmatrix}\right] \rightarrow a\cdot c + b\cdot d$

Vector outer product: $\begin{pmatrix}a\\ b\end{pmatrix}\cdot\begin{pmatrix}c & d\end{pmatrix} \rightarrow \begin{pmatrix}a\cdot c & a\cdot d\\ b\cdot c & b\cdot d\end{pmatrix}$

Matrix-vector product: $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} \rightarrow \begin{pmatrix}a\cdot x + b\cdot y\\ c\cdot x + d\cdot y\end{pmatrix} \qquad \begin{pmatrix}x & y\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{T} \rightarrow \begin{pmatrix}a\cdot x + b\cdot y & c\cdot x + d\cdot y\end{pmatrix}$

Expectation value: $\begin{pmatrix}x & y\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} \text{ simplify } \rightarrow a\cdot x^{2} + d\cdot y^{2} + b\cdot x\cdot y + c\cdot x\cdot y$

$$\begin{pmatrix}x & y\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{T}\cdot\begin{pmatrix}x\\ y\end{pmatrix} \text{ simplify } \rightarrow a\cdot x^{2} + d\cdot y^{2} + b\cdot x\cdot y + c\cdot x\cdot y$$

$$\mathrm{tr}\left[\begin{pmatrix}x\\ y\end{pmatrix}\cdot\begin{pmatrix}x & y\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right] \rightarrow a\cdot x^{2} + d\cdot y^{2} + b\cdot x\cdot y + c\cdot x\cdot y$$

Matrix product: $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}w & x\\ y & z\end{pmatrix} \rightarrow \begin{pmatrix}a\cdot w + b\cdot y & a\cdot x + b\cdot z\\ c\cdot w + d\cdot y & c\cdot x + d\cdot z\end{pmatrix}$

Vector tensor product: $\begin{pmatrix}a\\ b\end{pmatrix}\otimes\begin{pmatrix}c\\ d\end{pmatrix} = \begin{pmatrix}ac\\ ad\\ bc\\ bd\end{pmatrix}$

Matrix tensor product: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} w & x \\ y & z \end{pmatrix} & b\begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ c\begin{pmatrix} w & x \\ y & z \end{pmatrix} & d\begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

Tensor multiplication of several of the spin operators using Mathcad's **kronecker** command:

$$\mathrm{kronecker}(S_z, S_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{kronecker}(S_x, S_x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{kronecker}(S_d, S_d) = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix} \qquad \mathrm{kronecker}(S_z, S_x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathrm{kronecker}(S_z, S_d) = \begin{pmatrix} 0.707 & 0.707 & 0 & 0 \\ 0.707 & -0.707 & 0 & 0 \\ 0 & 0 & -0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 \end{pmatrix} \qquad \mathrm{kronecker}(S_z, I) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathrm{kronecker}(S_x, I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathrm{kronecker}(S_d, I) = \begin{pmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 0.707 & 0 & 0.707 \\ 0.707 & 0 & -0.707 & 0 \\ 0 & 0.707 & 0 & -0.707 \end{pmatrix}$$

Mathcad's **kronecker** command is only useful for matrix tensor multiplication, but can be adapted to carry out vector tensor multiplication in the manner shown below. Two matrices are created using the null vector, tensor multiplied and everything but the first column of the product matrix is discarded. The singlet spin state is formed using this method in the S_x, S_z and S_d spin bases.

Null vector: $$N := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Psi(a,b) := \mathrm{submatrix}(\mathrm{kronecker}(\mathrm{augment}(a,N), \mathrm{augment}(b,N)), 1, 4, 1, 1)$$

$$\frac{\Psi(S_{zu}, S_{zd}) - \Psi(S_{zd}, S_{zu})}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix} \qquad \frac{\Psi(S_{xd}, S_{xu}) - \Psi(S_{xu}, S_{xd})}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

$$S_{du} := \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \qquad S_{dd} := \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \qquad \frac{\Psi(S_{du}, S_{dd}) - \Psi(S_{dd}, S_{du})}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$