Density Matrix, Bloch Vector and Entropy

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A spin-1/2 state is represented by the following density matrix. $\rho := \begin{pmatrix} \frac{2}{3} & \frac{1}{6} - \frac{i}{3} \\ \frac{1}{6} + \frac{i}{3} & \frac{1}{3} \end{pmatrix}$

Show that this is a mixed state. $\operatorname{tr}(\rho) \rightarrow 1$ $\operatorname{tr}(\rho^2) \rightarrow \frac{5}{6}$

A value of less than unity for the trace of the square of the density matrix indicates a mixed state.

Given the Pauli matrices, $\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

calculate the Bloch vector and its magnitude.

$$R_x := \operatorname{tr}(\rho\cdot\sigma_x) \rightarrow \frac{1}{3} \qquad R_y := \operatorname{tr}(\rho\cdot\sigma_y) \rightarrow \frac{2}{3} \qquad R_z := \operatorname{tr}(\rho\cdot\sigma_z) \rightarrow \frac{1}{3} \qquad R := \sqrt{R_x{}^2 + R_y{}^2 + R_z{}^2} = 0.816$$

A value of less than unity for the magnitude of the Bloch vector also indicates a mixed state.

Calculate the eigenvalues, λ, of ρ and use them to calculate the entropy of the state represented by ρ.

$$S(\rho) = -\operatorname{Tr}(\rho\cdot\ln(\rho)) = -\sum_i \left(\lambda_i\cdot\log(\lambda_i, 2)\right) = \log\left[\prod_i (\lambda_i)^{-\lambda_i}, 2\right]$$

$$\lambda := \operatorname{eigenvals}(\rho) = \begin{pmatrix} 0.908 \\ 0.092 \end{pmatrix} \qquad -\sum_{i=1}^{2} \left(\lambda_i\cdot\log(\lambda_i, 2)\right) = 0.442 \qquad \log\left[\prod_{i=1}^{2} (\lambda_i)^{-\lambda_i}, 2\right] = 0.442$$

Use the magnitude of the Bloch vector to calculate the entropy (Haroche and Raimond, p. 168).

Eigenvalues: $\frac{1+R}{2} = 0.908$ $\frac{1-R}{2} = 0.092$ $-\frac{1+R}{2}\cdot\log\left(\frac{1+R}{2}, 2\right) - \frac{1-R}{2}\cdot\log\left(\frac{1-R}{2}, 2\right) = 0.442$

Calculate the expectation values for the measurements of spin in the x- and z-directions.

$$\operatorname{tr}(\rho\cdot\sigma_x) \rightarrow \frac{1}{3} \qquad \operatorname{tr}(\rho\cdot\sigma_z) \rightarrow \frac{1}{3}$$

$$Sxu := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad Sxd := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad Szu := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Szd := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Form projection operators for spin-up and spin-down in the x- and z-directions using the spin states provided above, and calculate the probabilities for spin-up and spin-down measurements in each direction. The eigenvalue for Sxu and Szu is +1 and the eigenvalue for Sxd and Szd is -1. It is easy to see that the results below are consistent with the expectation value calculations.

$$Sxu\cdot Sxu^T \rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad Sxd\cdot Sxd^T \rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad Szu\cdot Szu^T \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Szd\cdot Szd^T \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$tr\left(Sxd\cdot Sxd^T\cdot\rho\right) \rightarrow \frac{1}{3} \qquad tr\left(Sxu\cdot Sxu^T\cdot\rho\right) \rightarrow \frac{2}{3} \qquad tr\left(Szd\cdot Szd^T\cdot\rho\right) \rightarrow \frac{1}{3} \qquad tr\left(Szu\cdot Szu^T\cdot\rho\right) \rightarrow \frac{2}{3}$$

$$Sxd^T\cdot\rho\cdot Sxd \rightarrow \frac{1}{3} \qquad Sxu^T\cdot\rho\cdot Sxu \rightarrow \frac{2}{3} \qquad Szd^T\cdot\rho\cdot Szd \rightarrow \frac{1}{3} \qquad Szu^T\cdot\rho\cdot Szu \rightarrow \frac{2}{3}$$

Calculate the eigenvectors of the density matrix and use them to diagonalize it, showing that the diagonal elements are the eigenvalues as calculated above.

$$Vecs := eigenvecs(\rho) \qquad Vecs = \begin{pmatrix} 0.839 & -0.243 + 0.487i \\ 0.243 + 0.487i & 0.839 \end{pmatrix}$$

$$\left(\overline{Vecs}\right)^T\cdot\rho\cdot Vecs = \begin{pmatrix} 0.908 & 0 \\ 0 & 0.092 \end{pmatrix} \qquad Vecs^{-1}\cdot\rho\cdot Vecs = \begin{pmatrix} 0.908 & 0 \\ 0 & 0.092 \end{pmatrix}$$

Using the density matrix of the pure state Sxu show that its entropy is 0.

$$\rho := \frac{1}{2}\cdot\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Show that this is a pure state. $\quad tr(\rho) \rightarrow 1 \qquad tr\left(\rho^2\right) \rightarrow 1$

Calculate the Bloch vector and its magnitude.

$$R_x := tr\left(\rho\cdot\sigma_x\right) \rightarrow 1 \qquad R_y := tr\left(\rho\cdot\sigma_y\right) \rightarrow 0 \qquad R_z := tr\left(\rho\cdot\sigma_z\right) \rightarrow 0 \qquad R := \sqrt{R_x^{\,2} + R_y^{\,2} + R_z^{\,2}} = 1$$

A value of unity for the magnitude of the Bloch vector indicates a pure state.

Calculate the eigenvalues of ρ and use them to calculate the entropy of the state represented by ρ.

$$\lambda := eigenvals(\rho) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad -\sum_{i=1}^{2}\left(\lambda_i\cdot\log\left(\lambda_i, 2\right)\right) = 0 \qquad \log\left[\prod_{i=1}^{2}\left(\lambda_i\right)^{-\lambda_i}, 2\right] = 0$$

Two-particle, entangled Bell states are also pure states. This is demonstrated for the anti-symmetric singlet state, Ψ_m.

$$\left|\Psi_m\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_1\right\rangle\left|\downarrow_2\right\rangle - \left|\downarrow_1\right\rangle\left|\uparrow_2\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Psi_m := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \rho := \Psi_m \cdot \Psi_m^{\,T} \qquad \rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{tr}(\rho) = 1 \qquad \mathrm{tr}\left(\rho^2\right) = 1$$

One of the eigenvalues of ρ is 1 and the rest are 0, so the entropy is 0.

$$\lambda := \mathrm{eigenvals}(\rho) \qquad \lambda = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \log\left[\prod_{i=1}^{4} (\lambda_i)^{-\lambda_i}, 2\right] = 0$$

When a two-spin system is in an entangled superposition such as Ψ_m, the individual spins states are not definite. Tracing (see below) the composite density matrix over spin 2 yields the following spin density matrix for spin 1 showing that it is a mixed state with one unit of entropy.

$$\rho_1 := \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T + \frac{1}{2} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \rightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad \mathrm{tr}\left(\rho_1\right) \rightarrow 1 \qquad \mathrm{tr}\left(\rho_1^{\,2}\right) \rightarrow \frac{1}{2}$$

$$\lambda := \mathrm{eigenvals}\left(\rho_1\right) = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \qquad -\sum_{i=1}^{2} \left(\lambda_i \cdot \log\left(\lambda_i, 2\right)\right) = 1 \qquad \log\left[\prod_{i=1}^{2} (\lambda_i)^{-\lambda_i}, 2\right] = 1$$

The same result holds for spin 2.

In this entangled state we know the state of the composite system (zero entropy), but do not know the states of the individual spins (two units of entropy).

The trace operation is outlined below:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle\right]$$

$$\widehat{\rho_{12}} = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left[|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle\right]\left[\langle\downarrow_2|\langle\uparrow_1| - \langle\uparrow_2|\langle\downarrow_1|\right] = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\widehat{\rho_1} = \frac{1}{2}\left[\langle\uparrow_2|\Psi\rangle\langle\Psi|\uparrow_2\rangle + \langle\downarrow_2|\Psi\rangle\langle\Psi|\downarrow_2\rangle\right] = \frac{1}{2}\left[|\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_1\rangle\langle\uparrow_1|\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widehat{\rho_2} = \frac{1}{2}\left[\langle\uparrow_1|\Psi\rangle\langle\Psi|\uparrow_1\rangle + \langle\downarrow_1|\Psi\rangle\langle\Psi|\downarrow_1\rangle\right] = \frac{1}{2}\left[|\downarrow_2\rangle\langle\downarrow_2| + |\uparrow_2\rangle\langle\uparrow_2|\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$