# Quantum Entanglement Leads to Nonclassical Correlations

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This tutorial employs a tensor algebra approach to a *gedanken* experiment published by P. K. Aravind in the 2004 October issue of the *American Journal of Physics*. Aravind's thought experiment demonstrates how quantum entanglement leads directly to bizarre nonclassical correlations.

A source emits the following four-particle entangled state, with particles 1 and 3 going to Alice and particles 2 and 4 going to Bob. $\alpha$ and $\beta$ are the eigenstates of the Pauli $\sigma_z$ operator.

$$\alpha := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \beta := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\beta\rangle_2\right) \otimes \frac{1}{\sqrt{2}}\left(|\alpha\rangle_3|\alpha\rangle_4 + |\beta\rangle_3|\beta\rangle_4\right)$$

$$= \frac{1}{2}\left(|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3|\alpha\rangle_4 + |\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3|\beta\rangle_4 + |\beta\rangle_1|\beta\rangle_2|\alpha\rangle_3|\alpha\rangle_4 + |\beta\rangle_1|\beta\rangle_2|\beta\rangle_3|\beta\rangle_4\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}^T$$

$$\Psi := \frac{1}{2}\cdot\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}^T$$

Alice and Bob each have six measurement choices: R1, R2, R3, C1, C2, and C3. Each choice consists of a sequence of three measurements on the entangled spin pair they receive. These are shown in the table below. A sequence of measurements is possible because the operators in each row and each column mutually commute, as will be shown later.

| | C1 | C2 | C3 |
|---|---|---|---|
| R1 | $I \otimes \sigma_z$ | $\sigma_z \otimes I$ | $\sigma_z \otimes \sigma_z$ |
| R2 | $\sigma_X \otimes I$ | $I \otimes \sigma_x$ | $\sigma_x \otimes \sigma_x$ |
| R3 | $\sigma_x \otimes \sigma_z$ | $\sigma_z \otimes \sigma_x$ | $\sigma_y \otimes \sigma_y$ |

Alice and Bob independently and randomly set their detectors to one of the six possible settings each time the source emits the entangled particles, and record the result (+1 or -1) for each panel. After a statistically meaningful number of events they compare their results.

The operators required for this exercise are as follows:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Alice and Bob's measurement operators are constructed in tensor format:

$$A(a,b,c,d) := \text{kronecker}(a,\text{kronecker}(b,\text{kronecker}(c,d))) \qquad B(a,b,c,d) := \text{kronecker}(a,\text{kronecker}(b,\text{kronecker}(c,d)))$$

Because Alice gets particles 1 and 3, and Bob particles 2 and 4, their measurement operators in tensor notation are as shown below.

$$\text{Alice}\quad \begin{pmatrix} A(I,I,\sigma_z,I) & A(\sigma_z,I,I,I) & A(\sigma_z,I,\sigma_z,I) \\ A(\sigma_x,I,I,I) & A(I,I,\sigma_x,I) & A(\sigma_x,I,\sigma_x,I) \\ A(\sigma_x,I,\sigma_z,I) & A(\sigma_z,I,\sigma_x,I) & A(\sigma_y,I,\sigma_y,I) \end{pmatrix} \qquad \text{Bob}\quad \begin{pmatrix} B(I,I,I,\sigma_z) & B(I,\sigma_z,I,I) & B(I,\sigma_z,I,\sigma_z) \\ B(I,\sigma_x,I,I) & B(I,I,I,\sigma_x) & B(I,\sigma_x,I,\sigma_x) \\ B(I,\sigma_x,I,\sigma_z) & B(I,\sigma_z,I,\sigma_x) & B(I,\sigma_y,I,\sigma_y) \end{pmatrix}$$

Where $A(I,I,\sigma_z,I)$ stands for $I \otimes I \otimes \sigma_z \otimes I$ and means Alice's operator for her particle is $I \otimes \sigma_z$ .

Using a representative row and column, we show that the measurement operators in the rows and columns of the measurement grid commute.

First row:
$$\begin{pmatrix} A(I,I,\sigma_z,I)\cdot A(\sigma_z,I,I,I) - A(\sigma_z,I,I,I)\cdot A(I,I,\sigma_z,I) \\ A(I,I,\sigma_z,I)\cdot A(\sigma_z,I,\sigma_z,I) - A(\sigma_z,I,\sigma_z,I)\cdot A(I,I,\sigma_z,I) \\ A(\sigma_z,I,I,I)\cdot A(\sigma_z,I,\sigma_z,I) - A(\sigma_z,I,\sigma_z,I)\cdot A(\sigma_z,I,I,I) \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Third column:
$$\begin{pmatrix} A(\sigma_z,I,\sigma_z,I)\cdot A(\sigma_x,I,\sigma_x,I) - A(\sigma_x,I,\sigma_x,I)\cdot A(\sigma_z,I,\sigma_z,I) \\ A(\sigma_z,I,\sigma_z,I)\cdot A(\sigma_y,I,\sigma_y,I) - A(\sigma_y,I,\sigma_y,I)\cdot A(\sigma_z,I,\sigma_z,I) \\ A(\sigma_x,I,\sigma_x,I)\cdot A(\sigma_y,I,\sigma_y,I) - A(\sigma_y,I,\sigma_y,I)\cdot A(\sigma_x,I,\sigma_x,I) \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These results establish the validity of doing sequential measurements in any row or column.

The eigenvalues of the individual operators in each panel are +/-1, and for $\Psi$ the expectation values for the operators of the individual panels making up the rows and columns are zero. This is demonstrated for both Alice and Bob.

Alice
$$\begin{pmatrix} \Psi^T\cdot A(I,I,\sigma_z,I)\cdot\Psi & \Psi^T\cdot A(\sigma_z,I,I,I)\cdot\Psi & \Psi^T\cdot A(\sigma_z,I,\sigma_z,I)\cdot\Psi \\ \Psi^T\cdot A(\sigma_x,I,I,I)\cdot\Psi & \Psi^T\cdot A(I,I,\sigma_x,I)\cdot\Psi & \Psi^T\cdot A(\sigma_x,I,\sigma_x,I)\cdot\Psi \\ \Psi^T\cdot A(\sigma_x,I,\sigma_z,I)\cdot\Psi & \Psi^T\cdot A(\sigma_z,I,\sigma_x,I)\cdot\Psi & \Psi^T\cdot A(\sigma_y,I,\sigma_y,I)\cdot\Psi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Bob
$$\begin{pmatrix} \Psi^T\cdot B(I,I,I,\sigma_z)\cdot\Psi & \Psi^T\cdot B(I,\sigma_z,I,I)\,\Psi & \Psi^T\cdot B(I,\sigma_z,I,\sigma_z)\cdot\Psi \\ \Psi^T\cdot B(I,\sigma_x,I,I)\cdot\Psi & \Psi^T\cdot B(I,I,I,\sigma_x)\cdot\Psi & \Psi^T\cdot B(I,\sigma_x,I,\sigma_x)\cdot\Psi \\ \Psi^T\cdot B(I,\sigma_x,I,\sigma_z)\cdot\Psi & \Psi^T\cdot B(I,\sigma_z,I,\sigma_x)\cdot\Psi & \Psi^T\cdot B(I,\sigma_y,I,\sigma_y)\cdot\Psi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These calculations indicate that the individual panels on the measurement grids will flash +1 and -1 with equal frequency. However, if Alice and Bob measure the same observable pair, they always obtain the same eigenvalue, suggesting at this point a classical correlation between their individual results.

$$\begin{pmatrix} \Psi^T\cdot A(I,I,\sigma_z,I)\cdot B(I,I,I,\sigma_z)\cdot\Psi & \Psi^T\cdot A(\sigma_z,I,I,I)\cdot B(I,\sigma_z,I,I)\,\Psi & \Psi^T\cdot A(\sigma_z,I,\sigma_z,I)\cdot B(I,\sigma_z,I,\sigma_z)\cdot\Psi \\ \Psi^T\cdot A(\sigma_x,I,I,I)\cdot B(I,\sigma_x,I,I)\cdot\Psi & \Psi^T\cdot A(I,I,\sigma_x,I)\cdot B(I,I,I,\sigma_x)\cdot\Psi & \Psi^T\cdot A(\sigma_x,I,\sigma_x,I)\cdot B(I,\sigma_x,I,\sigma_x)\cdot\Psi \\ \Psi^T\cdot A(\sigma_x,I,\sigma_z,I)\cdot B(I,\sigma_x,I,\sigma_z)\cdot\Psi & \Psi^T\cdot A(\sigma_z,I,\sigma_x,I)\cdot B(I,\sigma_z,I,\sigma_x)\cdot\Psi & \Psi^T\cdot A(\sigma_y,I,\sigma_y,I)\cdot B(I,\sigma_y,I,\sigma_y)\cdot\Psi \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

This striking result appears to require that Alice and Bob's observables are "elements of reality" and therefore represent preexisting properties of their spin-1/2 particles. In other words, the particles carry instruction sets to their detectors which determine how the nine measurement panels respond. Not only that, it requires that the instruction sets for both detectors be identical.

However, the calculations below show that the expectation values for the sequence of measurements for rows 1, 2, 3, and columns 1 and 2 are +1. For column 3 the expectation value is -1.

Alice

Rows | Columns

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \begin{pmatrix} \Psi^T \cdot A(I,I,\sigma_z,I)\cdot A(\sigma_z,I,I,I)\cdot A(\sigma_z,I,\sigma_z,I)\cdot\Psi & \Psi^T \cdot A(I,I,\sigma_z,I)\cdot A(\sigma_x,I,I,I)\cdot A(\sigma_x,I,\sigma_z,I)\cdot\Psi \\ \Psi^T \cdot A(\sigma_x,I,I,I)\cdot A(I,I,\sigma_x,I)\cdot A(\sigma_x,I,\sigma_x,I)\cdot\Psi & \Psi^T \cdot A(\sigma_z,I,I,I)\cdot A(I,I,\sigma_x,I)\cdot A(\sigma_z,I,\sigma_x,I)\cdot\Psi \\ \Psi^T \cdot A(\sigma_x,I,\sigma_z,I)\cdot A(\sigma_z,I,\sigma_x,I)\cdot A(\sigma_y,I,\sigma_y,I)\cdot\Psi & \Psi^T \cdot A(\sigma_z,I,\sigma_z,I)\cdot A(\sigma_x,I,\sigma_x,I)\cdot A(\sigma_y,I,\sigma_y,I)\cdot\Psi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Bob

Rows | Columns

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \begin{pmatrix} \Psi^T \cdot B(I,I,I,\sigma_z)\cdot B(I,\sigma_z,I,I)\cdot B(I,\sigma_z,I,\sigma_z)\cdot\Psi & \Psi^T \cdot B(I,I,I,\sigma_z)\cdot B(I,\sigma_x,I,I)\cdot B(I,\sigma_x,I,\sigma_z)\cdot\Psi \\ \Psi^T \cdot B(I,\sigma_x,I,I)\cdot B(I,I,I,\sigma_x)\cdot B(I,\sigma_x,I,\sigma_x)\cdot\Psi & \Psi^T \cdot B(I,\sigma_z,I,I)\cdot B(I,I,I,\sigma_x)\cdot B(I,\sigma_z,I,\sigma_x)\cdot\Psi \\ \Psi^T \cdot B(I,\sigma_x,I,\sigma_z)\cdot B(I,\sigma_z,I,\sigma_x)\cdot B(I,\sigma_y,I,\sigma_y)\cdot\Psi & \Psi^T \cdot B(I,\sigma_z,I,\sigma_z)\cdot B(I,\sigma_x,I,\sigma_x)\cdot B(I,\sigma_y,I,\sigma_y)\cdot\Psi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

In order to validate a classical correlation between Alice and Bob's results based on the concept of "elements of reality", it is necessary to determine a set of instructions for the detectors that is in agreement with the highlighted results shown above. **Unfortunately this is impossible; there is no way to assign definite eigenvalues (+1 or -1) to each of the nine panels of the measurement grid that satisfy these results.**

> This contradiction shows that there is no solution to our puzzle based on instruction sets. A willingness to accept the notion of instruction sets (or "elements of reality") to begin with, followed by the recognition that they cannot provide a solution to our puzzle, amounts to an informal appreciation of the central point of Bell's theorem. P. K. Aravind, AJP 72, 1305 (2004).

On the basis of this *gedanken* experiment we must reject the position that quantum entities, quons (thank you Nick Herbert), have well-defined properties independent of measurement; in other words, that measurement simply reveals a preexisting state. Quantum entanglement is, as Schrödinger noted in 1935, "...the essential trait of the new theory, the one which forces a complete departure from all classical concepts."