Expressing Bell and GHZ States in Vector Format Using Mathcad

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Mathcad provides the *kronecker* command for matrix tensor multiplication. It requires square matrices for its arguments and therefore cannot be used directly for vector tensor multiplication. However, if a vector is augmented with a null vector (or matrix) to produce a square matrix, vector tensor multiplication can be carried out using *kronecker* and a *submatrix* command that discards everything except the first column of the product matrix. This technique is illustrated by putting the Bell and GHZ states in vector format.

The z- and x-direction spin eigenfunctions and the appropriate null vector are required.

$$z_u := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad z_d := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x_u := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x_d := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The Mathcad syntax for tensor multiplication of two 2-dimensional vectors.

$$\psi(a, b) := \text{submatrix}(\text{kronecker}(\text{augment}(a, n), \text{augment}(b, n)), 1, 4, 1, 1)$$

The four maximally entangled Bell states will be expressed in both the z- and the x-basis.

$$\left|\Phi_p\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_1\right\rangle\left|\uparrow_2\right\rangle + \left|\downarrow_1\right\rangle\left|\downarrow_2\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Phi_p := \frac{1}{\sqrt{2}} \cdot \left(\psi\left(z_u, z_u\right) + \psi\left(z_d, z_d\right)\right) \qquad \Phi_p = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \qquad \Phi_p := \frac{1}{\sqrt{2}} \cdot \left(\psi\left(x_u, x_u\right) + \psi\left(x_d, x_d\right)\right) \qquad \Phi_p = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix}$$

$$\left|\Phi_m\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_1\right\rangle\left|\uparrow_2\right\rangle - \left|\downarrow_1\right\rangle\left|\downarrow_2\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\Phi_m := \frac{1}{\sqrt{2}} \cdot \left(\psi\left(z_u, z_u\right) - \psi\left(z_d, z_d\right)\right) \qquad \Phi_m = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \qquad \Phi_m := \frac{1}{\sqrt{2}} \cdot \left(\psi\left(x_u, x_d\right) + \psi\left(x_d, x_u\right)\right) \qquad \Phi_m = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}$$

$$\left|\Psi_p\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_1\right\rangle\left|\downarrow_2\right\rangle + \left|\downarrow_1\right\rangle\left|\uparrow_2\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Psi_p := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_d\right) + \psi\left(z_d, z_u\right)\right) \quad \Psi_p = \begin{pmatrix} 0 \\ 0.707 \\ 0.707 \\ 0 \end{pmatrix}$$

$$\Psi_p := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(x_u, x_u\right) - \psi\left(x_d, x_d\right)\right) \quad \Psi_p = \begin{pmatrix} 0 \\ 0.707 \\ 0.707 \\ 0 \end{pmatrix}$$

$$\left|\Psi_m\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\uparrow_1\right\rangle\left|\downarrow_2\right\rangle - \left|\downarrow_1\right\rangle\left|\uparrow_2\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Psi_m := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_d\right) - \psi\left(z_d, z_u\right)\right) \quad \Psi_m = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

$$\Psi_m := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(x_d, x_u\right) - \psi\left(x_u, x_d\right)\right) \quad \Psi_m = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

The Mathcad syntax for tensor multiplication of three 2-dimensional vectors.

$$\psi(a, b, c) := \text{submatrix}(\text{kronecker}(\text{augment}(a, n), \text{kronecker}(\text{augment}(b, n), \text{augment}(c, n))), 1, 8, 1, 1)$$

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 \end{pmatrix}^T$$

$$\Psi_1 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_u, z_u\right) + \psi\left(z_d, z_d, z_d\right)\right) \qquad \Psi_1^T = \begin{pmatrix} 0.707 & 0 & 0 & 0 & 0 & 0 & 0 & 0.707 \end{pmatrix}$$

$$\Psi_2 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_u, z_u\right) - \psi\left(z_d, z_d, z_d\right)\right) \qquad \Psi_2^T = \begin{pmatrix} 0.707 & 0 & 0 & 0 & 0 & 0 & 0 & -0.707 \end{pmatrix}$$

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \pm 1 & 0 \end{pmatrix}^T$$

$$\Psi_3 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_u, z_d\right) + \psi\left(z_d, z_d, z_u\right)\right) \qquad \Psi_3^T = \begin{pmatrix} 0 & 0.707 & 0 & 0 & 0 & 0 & 0.707 & 0 \end{pmatrix}$$

$$\Psi_4 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_u, z_d\right) - \psi\left(z_d, z_d, z_u\right)\right) \qquad \Psi_4^T = \begin{pmatrix} 0 & 0.707 & 0 & 0 & 0 & 0 & -0.707 & 0 \end{pmatrix}$$

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \pm 1 & 0 & 0 \end{pmatrix}^T$$

$$\Psi_5 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_d, z_u\right) + \psi\left(z_d, z_u, z_d\right)\right) \qquad \Psi_5^T = \begin{pmatrix} 0 & 0 & 0.707 & 0 & 0 & 0.707 & 0 & 0 \end{pmatrix}$$

$$\Psi_6 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_d, z_u\right) - \psi\left(z_d, z_u, z_d\right)\right) \qquad \Psi_6^T = \begin{pmatrix} 0 & 0 & 0.707 & 0 & 0 & -0.707 & 0 & 0 \end{pmatrix}$$

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\pm\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0 & 1 & \pm 1 & 0 & 0 & 0\end{pmatrix}^{T}$$

$$\Psi_7 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_d, z_d\right) + \psi\left(z_d, z_u, z_u\right)\right) \qquad \Psi_7^{T} = \begin{pmatrix}0 & 0 & 0 & 0.707 & 0.707 & 0 & 0 & 0\end{pmatrix}$$

$$\Psi_8 := \frac{1}{\sqrt{2}}\cdot\left(\psi\left(z_u, z_d, z_d\right) - \psi\left(z_d, z_u, z_u\right)\right) \qquad \Psi_8^{T} = \begin{pmatrix}0 & 0 & 0 & 0.707 & -0.707 & 0 & 0 & 0\end{pmatrix}$$