Polarized Light and Quantum Mechanics: An Optical Analog of the Stern–Gerlach Experiment

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Abstract: Lecture demonstrations involving polarization of light are presented and discussed from the quantum theoretical point of view. The demonstrations clearly show that only two base states are required to describe the quantum state of a photon and they illustrate an optical analog of the Stern–Gerlach experiment that involved electrons.

Introduction

The classical theory of electromagnetic radiation offers an explanation of optical polarization phenomena. When speaking about photons, however, a quantum theory of light is required [1]. Simply put, a single photon possesses neither the electric nor magnetic fields associated with a classical beam of electromagnetic radiation. Photons, like electrons, are quantum entities; both are examples of particle-waves or quons [2]. Photons are peculiar quons, however. Just like electrons, photons possess an intrinsic angular momentum, but the photon is a relativistic quon. An electron possesses a rest mass, but a photon does not. For an electron, the intrinsic angular momentum is called spin and the Stern–Gerlach experiment established that to describe the spin state of an electron only two base states are required. In the language of quantum mechanics one says an electron possesses a spin angular momentum, quantum number s = 1/2, with the two spin base states given by $|m_s = 1/2\rangle$ (spin up) and $|m_s = -1/2\rangle$ (spin down). For a photon, on the other hand, the intrinsic angular momentum is often called spin or helicity, but could be called circular polarization. One says the photon possesses a polarization angular momentum quantum number j = 1 with the two polarization base states given by $|m_j = +1\rangle$ (right-hand circularly polarized) and $|m_j = -1\rangle$ (left-hand circularly polarized). What makes the photon peculiar is that the $m_j = 0$ state is missing due to the relativistic nature of the photon mentioned above. In other words, the photon's state of intrinsic angular momentum parallel to its direction of motion can only be +1 or –1 (in units of $\hbar$) and there is no component of intrinsic angular momentum perpendicular to the direction of motion.

Relativistic effects put aside, the bottom line is that only two base states, or basis functions for representation, are required to completely describe the polarization state of any photon. For circularly polarized light the base states could be designated conveniently as $|R\rangle$ and $|L\rangle$ instead of the $|+1\rangle$ and $|-1\rangle$ used above. Linearly polarized light can use $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ conveniently to designate the orthogonal vertical and horizontal polarization base states. One of the mysteries of quantum theory is that it makes no difference which base states we choose to describe the polarization quantum state provided they belong to a complete orthonormal set. That means we could describe the photon polarization basis states $|R\rangle$ and $|L\rangle$ in terms of the $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ basis, or vice versa. In the former case we can express the result as [1]:

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{h}\rangle + i|\mathrm{v}\rangle\right)$$
$$|L\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{h}\rangle - i|\mathrm{v}\rangle\right) \quad (1)$$

This convention is consistent with a right circularly polarized photon carrying an angular momentum of $\hbar$ in the direction of motion while a left circularly polarized photon carries an angular momentum of $-\hbar$ in the direction of motion.

Usually the choice of basis representation depends on experimental circumstance or interest. For the polarization experiments described below the orthogonal vertical- and horizontal-plane polarization base states $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ provide the most convenient basis.

In this article, we describe four simple lecture demonstrations involving plane-polarized light that provide a clear illustration of the two-base-state quantum nature of photon polarization, the interference of quantum probability amplitudes, and the optical analog of the famous Stern–Gerlach experiment.

We begin with the more familiar classical point of view. Classically speaking, a sheet of Polaroid film can produce plane-polarized light from an unpolarized light source. One says the electric field vector of the polarized beam of light is some vector that is perpendicular to the propagation direction of the beam and oriented at some angle θ with respect to some arbitrary reference. As is easily demonstrated, and well known,

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a second Polaroid sheet oriented at an angle $\theta \pm \pi/2$ with respect to the first sheet will completely block the light beam such that no light passes though the crossed-polarizers. Another well-known polarization phenomenon involves a double-refracting or birefringent material, like calcite, that will, classically speaking, separate a beam of unpolarized light into two beams of plane-polarized light that are polarized in mutually perpendicular directions. One beam is referred to as the ordinary ray while the second is termed the extraordinary ray. It is very easy to demonstrate this effect by simply viewing the double image of an object produced by a calcite crystal.

We would like to describe these classical demonstrations in the field of optics in terms of the quantum mechanics of a single photon. We can narrate the demonstrations using the language of quantum theory. Quantum theoretical descriptions of light are essential to the study of quantum chemistry and atomic/molecular spectroscopy. We use, in particular, a language and grammar of quantum theory developed by Dirac, usually referred to as Dirac's bra–ket notation, that has many useful applications in quantum chemistry [3].

Demonstration 1

Draw a narrow line in black ink, about 1 cm in length, on a sheet of transparency film and project the image of the line to a screen using an overhead projector. The light producing the image is unpolarized. Now place a Polaroid film over the transparency film. The light producing the filtered image of the line is now polarized and the intensity of the image is reduced by 50%. What does this have to do with quantum theory? The answer is that photons from the light bulb are unpolarized.

From the quantum point of view, unpolarized light, or even partially polarized light, consists of photons that cannot be described as in a definite polarization state. In a real sense, a photon assumes a known polarization state only after "doing the experiment" of measuring the photon polarization. For any direction in space a single photon is either entirely polarized in that direction or it is entirely polarized at right angles to that direction. There is no in-between here. This means that when the polarization state of a single photon is measured by a linear polarizer, such as the Polaroid sheet oriented in some definite direction in this demonstration, the linear polarization analyzer will either absorb the photon or allow the photon to pass through the analyzer.

The Polaroid measures the linear polarization of the photon. Assume the polarization axis of the Polaroid sheet defines the vertical position as a direction in space. If the photon passes through the linear polarization analyzer, then the photon polarization state is known definitely. It is polarized in the vertical direction. Only then can we describe the photon quantum state as the pure $|\mathrm{v}\rangle$ polarization state. If the polarizer absorbs the photon, the other possible outcome of measurement, it is polarized in the orthogonal horizontal direction and we can say the photon was in the pure $|\mathrm{h}\rangle$ polarization state. The Polaroid blocks the beam of photons in $|\mathrm{h}\rangle$ by absorption. Again, there are only two possible experimental results. The important point is that after the polarization measurement, the photon passing through the Polaroid is plane-polarized $|\mathrm{v}\rangle$. Is the photon polarized before measurement?

That is an interesting question. If 100% of the photons in a photon beam pass through the vertical polarizer then we know, only from this experiment, that all of the photons must have been plane-polarized in the $|\mathrm{v}\rangle$ polarization state. Similarly, if the vertical polarizer absorbs 100% of the photons then all of the photons must have been plane-polarized in the $|\mathrm{h}\rangle$ polarization state. In any other situation we simply have no way of knowing the polarization state of the photon before measurement and thus we cannot represent the photon by a definite quantum state. In other words, we cannot describe the photon as either $|\mathrm{v}\rangle$ or $|\mathrm{h}\rangle$.

Our ignorance of the polarization quantum state of any unpolarized photon before measurement is complete. In an unpolarized or even partially polarized photon beam the likelihood that the polarization state of a photon is either $|\mathrm{v}\rangle$ or $|\mathrm{h}\rangle$ is completely random. This randomness involves probabilities of the classical kind. If the probability that any photon is in the state $|\mathrm{v}\rangle$ or $|\mathrm{h}\rangle$ is 50–50, as in a classical coin toss, then we say that the photon beam is unpolarized. This is usually the case for photons emitted from an incandescent light bulb, and this is why the linear polarizer projects 50% of the light from the lamp in Demonstration 1. The difference between probabilities of the classical kind and probabilities of the quantum kind is very important and often presents a stumbling block to clear understanding of quantum theory. This difference is illustrated in the next demonstration.

Demonstration 2

Assume the Polaroid sheet in Demonstration 1 again defines the vertical position. Now simply place a second Polaroid sheet over the first. Observe that if the second Polaroid is oriented in the same vertical position as the first then there is no diminution of the intensity of the image of the line. If the second Polaroid is rotated by 90º to the horizontal position, however, the image is totally blocked. No photons pass through the crossed-polarizers. Of course, this is the familiar demonstration in the classical theory of light, but how does the quantum theory of photons describe the result?

In quantum theory, the base states are orthonormal, which for the choice of using plane-polarization base states $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ means that $\langle \mathrm{v}|\mathrm{v}\rangle = \langle \mathrm{h}|\mathrm{h}\rangle = 1$ and $\langle \mathrm{v}|\mathrm{h}\rangle = \langle \mathrm{h}|\mathrm{v}\rangle = 0$. We are now well into using Dirac notation and must be very clear what this notation means in the machinery of quantum theory.

In Dirac notation, $\langle \chi|\psi\rangle$ is a *quantum probability amplitude*, a pure number that may be complex. In our example if the photon is in the known polarization quantum state $|\psi\rangle$ and then has something done to it, $\langle \chi|\psi\rangle$ is the probability amplitude that the photon finds itself in polarization quantum state $|\chi\rangle$ as a result. The *probability* that the photon begins in $|\psi\rangle$ and ends up in $|\chi\rangle$ is given by the absolute square of the probability amplitude:

$$\text{Probability} = \langle \chi|\psi\rangle\langle \chi|\psi\rangle^* = \langle \chi|\psi\rangle\langle \psi|\chi\rangle = |\langle \chi|\psi\rangle|^2 \qquad (2)$$

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This is a fundamental maxim of the quantum theory. As we shall see, linear combinations of quantum probability amplitudes can lead to quantum interference effects.

We can use eq 2 to understand again why 50% of the photon beam from an incandescent bulb will pass through a vertical polarizing filter. We calculate the overall probability of finding the photon in $|\mathrm{v}\rangle$ after measuring the polarization with the vertical Polaroid filter:

$$\text{Probability} = \frac{1}{2}\left|\langle \mathrm{v}|\mathrm{v}\rangle\right|^2 + \frac{1}{2}\left|\langle \mathrm{v}|\mathrm{h}\rangle\right|^2 = \frac{1}{2}(1) + \frac{1}{2}(0) = \frac{1}{2} \quad (3)$$

Again, it must be emphasized that the factor of 1/2 in each of the two terms in eq 3 represents our assumption that the beam of unpolarized photons is a random 50–50 mixture of photons in the $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ polarization base states. [As long as the population of quantum states is completely random it does not matter which pair of base states we choose in eq 3. We obtain the same result if we choose as basis states $|R\rangle$ and $|L\rangle$ instead of $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$.] As a photon approaches the vertical polarizer we assume the likelihood that it is in the $|\mathrm{v}\rangle$ state is completely random with a probability 1/2. This assumption is confirmed by experiment in that 50% of the unpolarized photons pass through the vertical polarizer. Equation 3 is a mix of classical probability involving randomness and quantum probability involving probability amplitudes.

After the first Polaroid has sorted the unpolarized photon beam, all of the photons passing through the apparatus are in the pure $|\mathrm{v}\rangle$ polarization state. There is no random mixture of photons approaching the second Polaroid filter oriented in either the vertical or horizontal direction. For the horizontal orientation experiment, we can calculate the probability that a photon in the known $|\mathrm{v}\rangle$ state before the second, horizontal Polaroid will end up in the $|\mathrm{h}\rangle$ state after interacting with the horizontal Polaroid:

$$\text{Probability} = \left|\langle \mathrm{h}|\mathrm{v}\rangle\right|^2 = 0 \quad (4)$$

The calculation of quantum probability predicts that none of the vertically polarized photons will pass through the horizontal polarizing filter, as observed experimentally. If this were all we wanted to do then our introduction of Dirac notation might not seem worth the effort. With this background, however, now we can show something well worth the effort.

Demonstration 3

Take a third sheet of Polaroid filter material. (Polaroid sunglasses have been suggested for this purpose [4].) Place it over the first (vertical) and second (horizontal) polarizers and note that regardless of the orientation of the third polarizer no light passes through the combined apparatus. This is not surprising. There are no photons to interact with the third polarizer. Now put the third polarizer between the outer crossed-polarizers and rotate the middle polarizing filter. Surprise! Observe that now light does pass through the apparatus in spite of the cross-polarization of the outer two polarization filters. This polarization phenomenon is known as the three-polarizer paradox. It is a paradox because each polarizing filter acts to absorb light and one is left to explain how placing an additional absorber between the crossed-polarizers results in less absorbed light overall. The quantum explanation for this paradox is based on the superposition principle and the quantum interference of probability amplitudes [5].

By the superposition principle of quantum theory, any pure quantum state can be expressed as a linear combination of appropriate (i.e., complete and orthonormal) base states. What makes this interesting is that any number of possibilities exist for the base states representing the quon and each set is as good as any other. Until now, we have chosen, arbitrarily, $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ to be the two base states representing the polarization quantum state of a photon. This means that if the photon is in some other definite quantum state $|\psi\rangle$ then we can represent this state in the linear superposition

$$|\psi\rangle = |\mathrm{v}\rangle\langle \mathrm{v}|\psi\rangle + |\mathrm{h}\rangle\langle \mathrm{h}|\psi\rangle \quad (5)$$

where it is seen that the coefficients in the linear combination over base states are probability amplitudes. The superposition principle is another important maxim of quantum theory and often not well understood. The first thing that needs to be emphasized is that the photon in pure state $|\psi\rangle$ is *not a mixture* of photons in states $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$.

We now want to see what happens when we change base states. Earlier we noted that our vertical direction was rather arbitrarily chosen. Assume that another experimenter has a skewed view of things and this person's vertical is actually at some angle θ compared to ours. The situation is illustrated in Figure 1. What is the relationship of the skewed base states $|\mathrm{v}'\rangle$ and $|\mathrm{h}'\rangle$, in this rotated coordinate system, to the $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ base states we have chosen? To answer the question, express the skewed base states as the linear superposition of the original base states:

$$|\mathrm{v}'\rangle = |\mathrm{v}\rangle\langle \mathrm{v}|\mathrm{v}'\rangle + |\mathrm{h}\rangle\langle \mathrm{h}|\mathrm{v}'\rangle \quad (6)$$

and

$$|\mathrm{h}'\rangle = |\mathrm{v}\rangle\langle \mathrm{v}|\mathrm{h}'\rangle + |\mathrm{h}\rangle\langle \mathrm{h}|\mathrm{h}'\rangle \quad (7)$$

These are both versions of eq 5. The real question now is what are the probability amplitudes in eqs 6 and 7? Before answering the question there is a subtle but important point to make. When one changes base states one has to keep track of possible "phase changes." This is because all probability amplitudes really have a factor of $e^{i\delta}$ associated with them, where δ is called the phase angle. Values of δ may be quite arbitrarily chosen because the probabilities of any outcome do not depend on the value. One must be very consistent with one's choice, however. In the present case it is not difficult to show, using Figure 1, that the probability amplitudes can be taken as [1].

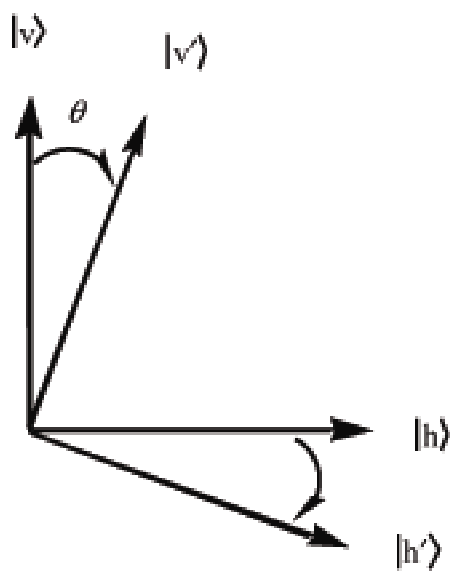

Figure 1. Coordinates showing the orthogonal horizontal and vertical directions in space perpendicular to the momentum of the photon. The figure also shows the sense of a positive, clockwise rotation about the direction of motion of the photon which is into the plane.

$$\begin{aligned}\langle v|v'\rangle &= \cos\theta \\ \langle h|v'\rangle &= \sin\theta \\ \langle v|h'\rangle &= -\sin\theta \\ \langle h|h'\rangle &= \cos\theta\end{aligned} \tag{8}$$

Although the amplitudes may be calculated for any chosen angle, take the case, for example, where θ equals 45º. In our reference frame we would call this a "diagonal" direction, but from the other person's skewed point of view it is "vertical." From eq 8, for θ equals 45º, we can write

$$\begin{aligned}|v'\rangle &= \frac{1}{\sqrt{2}}|v\rangle + \frac{1}{\sqrt{2}}|h\rangle \\ |h'\rangle &= -\frac{1}{\sqrt{2}}|v\rangle + \frac{1}{\sqrt{2}}|h\rangle\end{aligned} \tag{9}$$

The $|v'\rangle$ quantum state is not a 50–50 mixture of states $|v\rangle$ and $|h\rangle$ but it does have equal $|v\rangle$ character and $|h\rangle$ character. The same holds true for the $|h'\rangle$ state.

Now just as we can represent the skewed base states in terms of the original base states, we can do the inverse and represent the original base states in terms of the skewed basis. Again, the choice of basis for representing any photon polarization state is arbitrary. In the present case the inverse transformation of basis gives the following internally consistent amplitudes:

$$\begin{aligned}\langle v'|v\rangle &= \cos\theta \\ \langle v'|h\rangle &= \sin\theta \\ \langle h'|v\rangle &= -\sin\theta \\ \langle h'|h\rangle &= \cos\theta\end{aligned} \tag{10}$$

With these probability amplitudes in hand we can show how the quantum interference of amplitudes explains the three-polarizer paradox.

The first polarizer prepares a photon in the pure polarization state $|v\rangle$. Assume the second skewed polarizer is oriented at an angle θ with respect to the first. First, we express the pure state $|v\rangle$ as a superposition of the skewed base states $|v'\rangle$ and $|h'\rangle$, and in turn we represent the latter states by the original $|v\rangle$ and $|h\rangle$ base states. Using the probability amplitudes from eq 8 and eq 10 it looks like this:

$$\begin{aligned}|v\rangle &= \cos\theta|v'\rangle - \sin\theta|h'\rangle \\ &= \cos\theta\left(\cos\theta|v\rangle + \sin\theta|h\rangle\right) - \sin\theta\left(-\sin\theta|v\rangle + \cos\theta|h\rangle\right) \\ &= \left(\cos^2\theta + \sin^2\theta\right)|v\rangle + \left(\sin\theta\cos\theta - \sin\theta\cos\theta\right)|h\rangle \\ &= |v\rangle\end{aligned} \tag{11}$$

We seem to have gone to considerable length to prove an identity, but eq 11 contains what Richard Feynman [1] calls "the deep mystery of quantum mechanics—the interference of amplitudes." As eq 11 evidently demonstrates, the $|h\rangle$ character of $|v'\rangle$ destructively interferes with the $|h\rangle$ character of $|h'\rangle$ due to the *difference in phase* with which these amplitudes are added in the superposition. This destructive interference guarantees the pure-state nature of the $|v\rangle$ photon.

When the $|v\rangle$ photon from the first vertical polarizer interacts with the middle polarizer oriented in the skewed-vertical direction the $|h'\rangle$ character of the photon is removed by absorption and only the $|v'\rangle$ character passes the middle polarizer. In other words, the middle polarizer state selects photons for the pure $|v'\rangle$ state. In effect the absorption of $|h'\rangle$ by the middle polarizer removes one of the two interfering amplitudes in eq 11 and gives $|h\rangle$ character for those $|v'\rangle$ photons passing the middle filter. Because the $|v'\rangle$ photons have $|h\rangle$ character, the probability that a photon will pass through the third polarizer in the horizontal direction is given by $|\langle h|v'\rangle|^2$. Because this probability is not zero, photons of light pass through the third, horizontal polarizer and restore an image of the original line.

As mentioned, each of the three polarizers absorbs photons. Half of the photons from the light bulb are absorbed by the first vertical polarizer because we can consider the unpolarized light consisting of 50% photons in the $|v\rangle$ state, see eq 3. To quantify the total fraction of photons passing all three polarizers, take the example of the middle polarizer oriented at θ = 45º. Polarization measurements in this "diagonal" direction for a beam of photons known definitely to be in the $|\mathrm{v}\rangle$ state yield completely random results that are determined by the quantum probability amplitudes. Half of these $|\mathrm{v}\rangle$ photons are absorbed by the second diagonal polarizer. In other words, the probability that a $|\mathrm{v}\rangle$ photon will be found in the $|\mathrm{v}'\rangle$ state passing through the middle polarizer is given by the $\langle \mathrm{v}'|\mathrm{v}\rangle$ probability amplitude:

$$\text{Probability} = \left|\langle \mathrm{v}'|\mathrm{v}\rangle\right|^2 = \cos^2\theta = \cos^2\left(45^\circ\right) = \frac{1}{2} \qquad (12)$$

Finally, half of the $|\mathrm{v}'\rangle$ photons passing the middle polarizer are absorbed by the third horizontal polarizer because the probability that a $|\mathrm{v}'\rangle$ photon will be found in the $|\mathrm{h}\rangle$ state passing through the third polarizer is

$$\text{Probability} = \left|\langle \mathrm{h}|\mathrm{v}'\rangle\right|^2 = \sin^2\theta = \sin^2\left(45^\circ\right) = \frac{1}{2} \qquad (13)$$

This means that the fraction of photons from the light bulb that pass through the combined apparatus is (1/2)(1/2)(1/2) = 1/8 so the image of the line is dim, but it is definitely not zero.

Demonstration 4

Over the narrow line drawn on the transparency sheet above, place a calcite crystal. (Reasonably optical-quality calcite crystals of $3.0 \times 1.5 \times 1.5$ cm^3 in size can be purchased at very low cost in hobby stores with a mineral collection.) Observe the dual image of the line on a screen and rotate the crystal until the maximum separation of the two lines is made. The optical depth of the calcite crystal determines the magnitude of the maximum separation. For a crystal with a depth of 1.5 cm the two lines can be separated by about 2 mm. The calcite crystal is a different type of linear polarization analyzer than the Polaroid sheet. A single, unpolarized photon approaching the calcite crystal has the same 50–50 chance of becoming $|\mathrm{v}\rangle$ polarized or $|\mathrm{h}\rangle$ polarized as with the Polaroid sheet, but now both photon polarization states pass through the analyzer. All photons polarized along the optic axis of the crystal become $|\mathrm{v}\rangle$ polarized, say, while those polarized at right angles to the optic axis become $|\mathrm{h}\rangle$ polarized. The designation is arbitrary but the point is that two base states are required to describe linear polarization of the photons.

We can prove that the quantum state of the photons found in one beam is orthogonal to the quantum state of the photons in the second beam. Place a Polaroid sheet over the calcite crystal. Show that by rotating the Polaroid sheet over the dual image that a position can be found which completely filters one of the two images. The quantized direction in space of the Polaroid is now aligned with the quantized direction of the calcite crystal. After this position is noted, rotate the Polaroid sheet by 90º and observe the image situation is completely reversed. What was lost is found, and what was found is lost. This observation proves that the two images from the calcite crystal are polarized in mutually perpendicular directions, and that the polarization states of a photon require two, and only two, orthonormal base states. This experiment is an exact optical analog of the Stern–Gerlach experiment. The Stern–Gerlach apparatus can spatially separate a beam of unpolarized electrons into the two base states $|1/2\rangle$ and $|-1/2\rangle$ by the interaction of the electrons with an inhomogeneous magnetic field. The calcite crystal can spatially separate a beam of unpolarized photons into the two base states $|\mathrm{v}\rangle$ and $|\mathrm{h}\rangle$ by the interaction of the photons with an inhomogeneous crystal field. (A similar demonstration using a polarizing prism instead of a calcite crystal has been reported [6].)

These four lecture demonstrations are very easy to do. The explanations in terms of the quantum theory of photon angular momentum are involved, but the two-base-state nature of photon polarization allows for clear illustrations of conceptually difficult quantum effects.

References and Notes

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