Rudimentary Matrix Mechanics

Frank Rioux

A quon (an entity that exhibits both wave and particle aspects in the peculiar quantum manner - Nick Herbert, *Quantum Reality*, page 64) has a variety of properties each of which can take on two values. For example, it has the property of *hardness* and can be either *hard* or *soft*. It also has the property of *color* and can be either *black* or *white*, and the property of taste and be *sweet* or *sour*. The treatment that follows draws on material from Chapter 3 of David Z Albert's book, *Quantum Mechanics and Experience*.

The basic principles of matrix and vector math are provided in Appendix A. An examination of this material will demonstrate that most of the calculations presented in this tutorial can easily be performed without the aid of Mathcad or any other computer algebra program. In other words, they can be done by hand.

In the matrix formulation of quantum mechanics the hardness and color states are represented by the following vectors.

$$\text{Hard} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{Soft} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{Black} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{White} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Hard and *Soft* represent an **orthonormal basis** in the two-dimensional *Hardness* vector space.

$$\text{Hard}^T \cdot \text{Hard} = 1 \qquad \text{Soft}^T \cdot \text{Soft} = 1 \qquad \text{Hard}^T \cdot \text{Soft} = 0$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \qquad \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \qquad \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Likewise *Black* and *White* are an **orthonormal basis** in the two-dimensional *Color* vector space.

$$\text{Black}^T \cdot \text{Black} = 1 \qquad \text{White}^T \cdot \text{White} = 1 \qquad \text{Black}^T \cdot \text{White} = 0$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1 \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 1 \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0$$

The relationship between the two bases is reflected in the following projection calculations. Note: $\frac{1}{\sqrt{2}} = 0.707$

$$\text{Hard}^T \cdot \text{Black} = 0.707 \qquad \text{Hard}^T \cdot \text{White} = 0.707 \qquad \text{Soft}^T \cdot \text{Black} = 0.707 \qquad \text{Soft}^T \cdot \text{White} = -0.707$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0.707 \qquad \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0.707 \qquad \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0.707 \qquad \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = -0.707$$

The values calculated above are probability amplitudes. The absolute square of those values is the probability. In other words, the probability that a black quon will be found to be hard is 0.5. The probability that a white quon will be found to be soft is also 0.5.

$$\left(\left|\text{Hard}^T \cdot \text{Black}\right|\right)^2 = 0.5 \qquad \left(\left|\text{Hard}^T \cdot \text{White}\right|\right)^2 = 0.5 \qquad \left(\left|\text{Soft}^T \cdot \text{Black}\right|\right)^2 = 0.5 \qquad \left(\left|\text{Soft}^T \cdot \text{White}\right|\right)^2 = 0.5$$

$$\left[\left|(1\ \ 0)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\right|\right]^2 = 0.5 \qquad \left[\left|(1\ \ 0)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\right|\right]^2 = 0.5 \qquad \left[\left|(0\ \ 1)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\right|\right]^2 = 0.5 \qquad \left[\left|(0\ \ 1)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\right|\right]^2 = 0.5$$

Clearly *Black* and *White* can be written as **superpositions** of *Hard* and *Soft*, and vice versa.

$$\frac{1}{\sqrt{2}}\cdot(\text{Hard} + \text{Soft}) = \begin{pmatrix}0.707\\0.707\end{pmatrix} \qquad \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}0.707\\0.707\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot(\text{Hard} - \text{Soft}) = \begin{pmatrix}0.707\\-0.707\end{pmatrix} \qquad \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}0.707\\-0.707\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot(\text{Black} + \text{White}) = \begin{pmatrix}1\\0\end{pmatrix} \qquad \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\right] = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\cdot(\text{Black} - \text{White}) = \begin{pmatrix}0\\1\end{pmatrix} \qquad \frac{1}{\sqrt{2}}\cdot\left[\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} - \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\right] = \begin{pmatrix}0\\1\end{pmatrix}$$

Hard, *Soft*, *Black* and *White* are measurable properties and the vectors representing them are eigenstates of the *Hardness* and *Color* operators with **eigenvalues** +/- 1. The *Identity* operator is also given and will be discussed later. Of course, the *Hardness* and *Color* operators are just the Pauli spin operators in the z- and x-directions. Later the *Taste* operator will be introduced; it is the y-direction Pauli spin operator.

Operators

$$\text{Hardness} := \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \text{Color} := \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad \text{I} := \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Eigenvalue +1

$$\text{Hardness}\cdot\text{Hard} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\text{Color}\cdot\text{Black} = \begin{pmatrix}0.707\\0.707\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}0.707\\0.707\end{pmatrix}$$

Eigenvalue -1

$$\text{Hardness}\cdot\text{Soft} = \begin{pmatrix}0\\-1\end{pmatrix} \qquad \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$\text{Color}\cdot\text{White} = \begin{pmatrix}-0.707\\0.707\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}-0.707\\0.707\end{pmatrix}$$

Another way of showing this is by calculating the **expectation (or average) value**. Every time the hardness of a hard quon is measured the result is +1. Every time the hardness of a soft quon is measured the result is -1.

$$\text{Hard}^{T}\cdot\text{Hardness}\cdot\text{Hard} = 1 \qquad (1\ \ 0)\cdot\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$\text{Soft}^{T}\cdot\text{Hardness}\cdot\text{Soft} = -1 \qquad (0\ \ 1)\cdot\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = -1$$

$Black^T \cdot Color \cdot Black = 1$ $\quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1$ $\quad White^T \cdot Color \cdot White = -1$ $\quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = -1$

If a quon is in a state which is an eigenfunction of an operator, it means it has a well-defined value for the observable represented by the operator. If the quon is in a state which is not an eigenfunction of the operator, it does not have a well-defined value for the observable.

Hard and *Soft* are not **eigenfunctions** of the *Color* operator, and *Black* and *White* are not eigenfunctions of the *Hardness* operator. Hard and soft quons do not have a well-defined color, and black and white quons do not have a well-defined hardness.

$Hardness \cdot Black = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$ $\quad Hardness \cdot White = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$

$Color \cdot Hard = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad Color \cdot Soft = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Therefore their **expectation values** are zero. If the hardness of a black quon is measured, half the time it will register hard and half the time soft, because prior to observation the output state is a superposition of hard and soft.

$$Hardness \cdot Black = White = \frac{1}{\sqrt{2}} \cdot (Hard - Soft) \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

Similarly if the color of a soft quon is measured, half the time it will register black and half the time white. In summary, measurement **always** yields an eigenfunction of the measurement operator.

$$Color \cdot Soft = Hard = \frac{1}{\sqrt{2}} \cdot (Black + White) \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

$Black^T \cdot Hardness \cdot Black = 0$ $\quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$ $\quad White^T \cdot Hardness \cdot White = 0$ $\quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0$

$Hard^T \cdot Color \cdot Hard = 0$ $\quad \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ $\quad Soft^T \cdot Color \cdot Soft = 0$ $\quad \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

As the *Hardness-Color* commutator shows, the *Hardness* and *Color* operators do not commute. They represent incompatible observables; observables that cannot simultaneously have well-defined values.

$$Hardness \cdot Color - Color \cdot Hardness = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

This means that the measurement of the color and then the hardness of a hard quon gives a different result than the measurement of the hardness and then the color.

$Hardness \cdot Color \cdot Hard = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad Color \cdot Hardness \cdot Hard = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We can also look at this from the perspective of the uncertainty principle. The uncertainty in a measurement is the square root of the difference between the *mean of the square* and the *square of the mean*. Suppose we measure the color of a Black or White quon. Because Black and White are eigenfunctions of the Color operator the uncertainty in the measurement results are zero.

$$\sqrt{\text{Black}^T\cdot\text{Color}^2\cdot\text{Black}-\left(\text{Black}^T\cdot\text{Color}\cdot\text{Black}\right)^2}=0 \qquad \sqrt{\text{White}^T\cdot\text{Color}^2\cdot\text{White}-\left(\text{White}^T\cdot\text{Color}\cdot\text{White}\right)^2}=0$$

However, the measurement of the color of a Soft or Hard quon is by the same criterion uncertain.

$$\sqrt{\text{Soft}^T\cdot\text{Color}^2\cdot\text{Soft}-\left(\text{Soft}^T\cdot\text{Color}\cdot\text{Soft}\right)^2}=1 \qquad \sqrt{\text{Hard}^T\cdot\text{Color}^2\cdot\text{Hard}-\left(\text{Hard}^T\cdot\text{Color}\cdot\text{Hard}\right)^2}=1$$

The calculations of *Hardness* and *Color* reveal the strange behavior of quons. In the macro world we frequently find objects that simultaneously have well-defined values for these physical attributes. But we see this is not necessarily possible in the quantum world.

Mathcad has high-level commands which find the eigenvalues and eigenvectors of matrices (operators). Below it is shown that they give the same results as were demonstrated above.

$$\text{eigenvals}(\text{Hardness})=\begin{pmatrix}1\\-1\end{pmatrix} \qquad \text{eigenvec}(\text{Hardness},-1)=\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{eigenvec}(\text{Hardness},1)=\begin{pmatrix}1\\0\end{pmatrix}$$

$$\text{eigenvals}(\text{Color})=\begin{pmatrix}1\\-1\end{pmatrix} \qquad \text{eigenvec}(\text{Color},-1)=\begin{pmatrix}-0.707\\0.707\end{pmatrix} \qquad \text{eigenvec}(\text{Color},1)=\begin{pmatrix}0.707\\0.707\end{pmatrix}$$

Besides the properties of hardness and color, suppose the quon also has the property of taste, tasting either *Sweet* or *Sour*. The *Taste* operator is defined below and its eigenvalues and eigenvectors calculated.

Operator Eigenvalues Sweet/Sour Eigenvectors

$$\text{Taste}:=\begin{pmatrix}0&-i\\i&0\end{pmatrix} \qquad \text{eigenvals}(\text{Taste})=\begin{pmatrix}1\\-1\end{pmatrix} \qquad \text{eigenvecs}(\text{Taste})=\begin{pmatrix}-0.707i&0.707\\0.707&-0.707i\end{pmatrix} \qquad \text{Sw}:=\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}-i\\1\end{pmatrix} \qquad \text{So}:=\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-i\end{pmatrix}$$

Squaring the *Hardness*, *Color* and *Taste* operators gives the *Identity* operator, that is they are unitary matrices. The *Identity* operator leaves the vector it operates on unchanged.

$$\text{Hardness}^2=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \text{Color}^2=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\text{Taste}^2=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Another important property of these operators is that they are equal to their Hermitian conjugate as shown below. The physical significance of this is that they have real eigenvalues, something we know from earlier calculations.

$$\left(\overline{\text{Hardness}}\right)^T=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \left[\overline{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\right]^T=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \left(\overline{\text{Color}}\right)^T=\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \left[\overline{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\right]^T=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\left(\overline{\text{Taste}}\right)^{T} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \left[\overline{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}\right]^{T} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The Hadamard matrix is another operator which is important in quantum optics and quantum computing.

$$\text{Hadamard} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Hadamard matrix performs a Fourier transform between the *Hardness* and *Color* basis vectors.

$$\text{Hadamard} \cdot \text{Hard} = \text{Black} \qquad \text{Hadamard} \cdot \text{Hard} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \text{Hadamard} \cdot \text{Black} = \text{Hard} \qquad \text{Hadamard} \cdot \text{Black} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{Hadamard} \cdot \text{Soft} = \text{White} \qquad \text{Hadamard} \cdot \text{Soft} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \qquad \text{Hadamard} \cdot \text{White} = \text{Soft} \qquad \text{Hadamard} \cdot \text{White} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenvalues and eigenvectors of the Hadamard matrix:

$$\text{eigenvals}(\text{Hadamard}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{eigenvec}(\text{Hadamard}, -1) = \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \qquad \text{eigenvec}(\text{Hadamard}, 1) = \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix}$$

The Hadamard matrix is also unitary and its own Hermitian conjugate like the other matrices.

$$\text{Hadamard}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \left(\overline{\text{Hadamard}}\right)^{T} = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

Composite Systems

Quantum mechanics gets even more interesting for composite systems - quantum systems consisting of two or more quons. Suppose two quons are created in the same event and one is hard (**H**) and the other is soft (**S**), but of course because of the indistinquishability principle we don't know which is which. Under this circumstance an appropriate state vector is the following *entangled superposition*. (See Appendix A for vector tensor multiplication),

$$\Psi = \frac{1}{\sqrt{2}}\left[\left|H\right\rangle_A \left|S\right\rangle_B - \left|S\right\rangle_A \left|H\right\rangle_B\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

If the hardness of A or the hardness of B is measured the expectation value is 0, because half the time the quon will be found to be hard (+1) and half the time soft (-1). However if the hardness of both quons is measured the joint expectation value is -1, because they are in opposite hardness states. This is perfect anti-correlation. The joint measurements show correlation in spite of the fact that the individual measurements are totally random. This is the "spooky action at a distance" that bothered Einstein. Kronecker is Mathcad's command for matrix tensor multiplication which is illustrated in Appendix A.

$$\Psi^{T} \cdot \text{kronecker}(\text{Hardness}, I) \cdot \Psi = 0 \qquad \Psi^{T} \cdot \text{kronecker}(I, \text{Hardness}) \cdot \Psi = 0 \qquad \Psi^{T} \cdot \text{kronecker}(\text{Hardness}, \text{Hardness}) \cdot \Psi = -1$$

Now suppose we do color measurements on the same pair of quons. Again we find perfect color anti-correlation between the two quons. Individually the color measurements are randomly black (**B**) or white (**W**), but when we measure the color of both quons we always get different colors.

$\Psi^T \cdot \text{kronecker}(\text{Color}, I) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(I, \text{Color}) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Color}) \cdot \Psi = -1$

This result can be understood by recalling that black and white are superpositions of hard and soft. Substitution of the appropriate superpositions into the original composite state vector expresses it in the black/white basis and reveals the perfect anti-correlation.

$$\Psi = \frac{1}{\sqrt{2}}\left[\left|W\right\rangle_A \left|B\right\rangle_B - \left|B\right\rangle_A \left|W\right\rangle_B\right] = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

The same thing that is true for black and white is also true for sweet (**Sw**) and sour (**So**). The taste measurements are individually random, but collectively perfectly anti-correlated.

$\Psi^T \cdot \text{kronecker}(\text{Taste}, I) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(I, \text{Taste}) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(\text{Taste}, \text{Taste}) \cdot \Psi = -1$

Below the original state vector is written in the sweet/sour basis.

$$\Psi = \frac{1}{\sqrt{2}}\left[\left|So\right\rangle_A \left|Sw\right\rangle_B - \left|Sw\right\rangle_A \left|So\right\rangle_B\right] = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

If different properties of the quons are measured the expectation values are zero - there is no correlation.

$\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Color}) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Taste}) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Taste}) \cdot \Psi = 0$

A realist believes that objects, macro, micro or nano, have well-defined properties prior to and independent of the nature of the observation performed on them. Experiment simply reveals the unknown state of the system under observation.

Objects with three properties (hardness, color and taste) which can be in either of two states (hard +1, soft -1, black +1, white -1, sweet +1 and sour -1) can be in any one of eight states according to a realist: HBSw, HBSo, HWSw, HWSo, SBSw, SBSo, SWSw and SWSo. Due to the correlation values when the same property is measured on both quons, the realist can explain all measurement results as shown in the table below.

Because the states were constructed on the basis of anti-correlation for hardness, color and taste, it is only necessary to show agreement between the quantum calculations and the realist's states for the cases in which different properties are measured. The three right-hand columns of the table show no correlation, in agreement with the quantum calculations.

QuonA	QuonB	Hardness-Color	Hardness-Taste	Color-Taste
HBSw	SWSo	1 x -1= -1	1 x -1= -1	1 x -1= -1
HBSo	SWSw	1 x -1= -1	1 x 1= 1	1 x 1= 1
HWSw	SBSo	1 x 1= 1	1 x -1= -1	-1 x -1= 1
HWSo	SBSw	1 x 1= 1	1 x 1= 1	-1 x 1= -1
Expectation	Value	0	0	0

In spite of this agreement, the quantum theorist objects. Earlier it was shown that the hardness and color operators do not commute, meaning that from the quantum perspective hardness and color cannot be simultaneously well-defined. The same is true for hardness and taste, and for color and taste. Therefore, the states in the table proposed by the realist are not legitimate.

$$\text{Hardness}\cdot\text{Taste} - \text{Taste}\cdot\text{Hardness} = \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} \qquad \text{Color}\cdot\text{Taste} - \text{Taste}\cdot\text{Color} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

The superpositions tell the same story. For example, if a quon is hard (H) its color and taste are indeterminate because hard is an even superposition of black and white, and sweet and sour.

$$H = \frac{1}{\sqrt{2}}\cdot(B + W) \qquad H = \frac{1}{\sqrt{2}}\cdot(i\cdot Sw + So)$$

While this line of reasoning is compelling, the realist is undeterred. The fact that quantum mechanics can't assign specific values to all properties of an object is evidence that it does not provide a complete theory of reality!

Thought experiments like this clarify the conflict between quantum theory and local realism, but they do not provide, as we have seen, a final adjudication of the disagreement. That changed in 1964 with a theoretical analysis by John Bell that showed that there are experimental situations where the predictions of quantum mechanics and local realism are in disagreement. We look at one of them now.

Odor is another physical property of objects. The Hadamard operator is renamed the Odor operator. It has two eigenstates **P**leasant and **F**oul, with eigenvalues +1 and -1, respectively, as shown below.

$$\text{Odor} := \text{Hadamard} \qquad \text{eigenvals(Odor)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad P := \text{eigenvec(Odor}, 1) = \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \qquad F := \text{eigenvec(Odor}, -1) = \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}\left[|P\rangle_A |F\rangle_B - |F\rangle_A |P\rangle_B\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \otimes \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} - \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \otimes \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Carrying out some of the same quantum mechanical calculations that we have done for the other observable properties, we see that the individual odor measurements are random, there is perfect anti-correlation in the joint odor measurements and intermediate anti-correlation in the joint hardness-odor measurements. This latter result is of utmost importance, because a local realist can't explain it.

$$\Psi^T\cdot\text{kronecker(Odor}, I)\cdot\Psi = 0 \qquad \Psi^T\cdot\text{kronecker(Odor}, \text{Odor})\cdot\Psi = -1 \qquad \Psi^T\cdot\text{kronecker(Hardness}, \text{Odor})\cdot\Psi = -0.707$$

Given that we are now dealing with four properties, each of which can have two values, there are 16 composite states. However, for now we only need to consider the four states involving the properites of hardness and odor to show that the local realist model cannot explain the anti-correlation predicted by quantum mechanics for the joint hardness-odor measurements. In the following table H =Hard, S = Soft, P = Pleasant, and F = Foul. Appendix B provides a complete analysis for all four observable properties.

Quon1	Quon2	HardnessHardness	OdorOdor	HardnessOdor
HP	SF	−1	−1	−1
HF	SP	−1	−1	1
SP	HF	−1	−1	1
SF	HP	−1	−1	−1
Expectation	Value	−1	−1	0
Quantum	Result	−1	−1	−0.707

The last column shows that a local realist model predicts no correlation between the joint hardness-odor measurements, while a quantum mechanical calculation predicts anti-correlation of -0.707. This example illustrates the significance of Bell's analysis: there are experiments for which a local realist model cannot reproduce all the quantum mechanical predictions. And to date the quantum mechanical predictions have been verified experimentally. Thus, a local realist model of nature is not tenable. As mentioned above Appendix B provides additional computational detail regarding this issue.

Concluding Remarks

The reason for using the properties of hardness, color, taste and odor in these exercises is to emphasize how different the quantum world is from the macro world that we occupy. It is not an uncommon experience (it has happened to me) to eat a piece of candy that is hard, white and sweet. But this is not possible for *quantum candy* because the matrix operators representing these observables do not commute. Therefore, the observables cannot simultaneously be well defined.

In quantum mechanics the hardness, color and taste operators,

$$\text{Hardness} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{Color} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{Taste} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

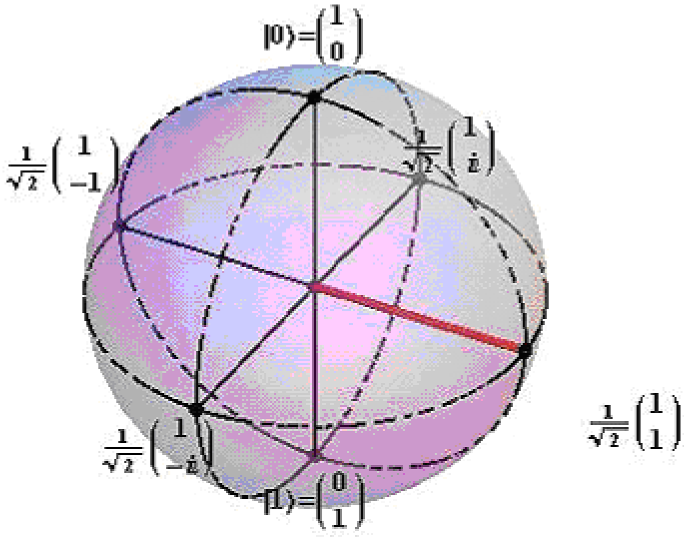
are actually the Pauli spin matrices and represent the observables for spin in the z-, x- and y-directions as mentioned earlier.

$$\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

They are also the operators for the rectilinear, diagonal and circular polarization properties of photons. In this case the eigenvectors are vertical, horizontal, diagonal, anti-diagonal, and right and left circular polarization.

$$V := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad H := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad D := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad A := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad R := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \quad L := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The geometrical relationship between these basis states is represented graphically on the Bloch sphere.



Appendix A: Vector and Matrix Math

Vector inner product: $(a\ \ b)\cdot\begin{pmatrix}c\\d\end{pmatrix} \rightarrow a\cdot c + b\cdot d$

Vector outer product: $\begin{pmatrix}c\\d\end{pmatrix}\cdot(a\ \ b) \rightarrow \begin{pmatrix}a\cdot c & b\cdot c\\ a\cdot d & b\cdot d\end{pmatrix}$ $\quad$ $tr\left[\begin{pmatrix}c\\d\end{pmatrix}\cdot(a\ \ b)\right] \rightarrow a\cdot c + b\cdot d$

Matrix-vector product: $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} \rightarrow \begin{pmatrix}a\cdot x + b\cdot y\\ c\cdot x + d\cdot y\end{pmatrix}$ $\quad$ $(x\ \ y)\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}^T \rightarrow (a\cdot x + b\cdot y\ \ \ c\cdot x + d\cdot y)$

Expectation value: $(x\ \ y)\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$ simplify $\rightarrow a\cdot x^2 + d\cdot y^2 + b\cdot x\cdot y + c\cdot x\cdot y$

$(x\ \ y)\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}^T\cdot\begin{pmatrix}x\\y\end{pmatrix}$ simplify $\rightarrow a\cdot x^2 + d\cdot y^2 + b\cdot x\cdot y + c\cdot x\cdot y$

$tr\left[\begin{pmatrix}x\\y\end{pmatrix}\cdot(x\ \ y)\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right] \rightarrow a\cdot x^2 + d\cdot y^2 + b\cdot x\cdot y + c\cdot x\cdot y$

Matrix product: $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}w & x\\ y & z\end{pmatrix} \rightarrow \begin{pmatrix}a\cdot w + b\cdot y & a\cdot x + b\cdot z\\ c\cdot w + d\cdot y & c\cdot x + d\cdot z\end{pmatrix}$

Vector tensor product: $$\begin{pmatrix}a\\b\end{pmatrix}\otimes\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}a\begin{pmatrix}c\\d\end{pmatrix}\\ b\begin{pmatrix}c\\d\end{pmatrix}\end{pmatrix} = \begin{pmatrix}ac\\ad\\bc\\bd\end{pmatrix}$$

Matrix tensor product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} w & x \\ y & z \end{pmatrix} & b\begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ c\begin{pmatrix} w & x \\ y & z \end{pmatrix} & d\begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

Matrix eigenvalues and eigenvectors (unnormalized):

$$\text{eigenvals}\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) \rightarrow \begin{pmatrix} a-b \\ a+b \end{pmatrix} \quad \text{or} \quad \left|\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}\right| = 0 \text{ solve}, \lambda \rightarrow \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix} \quad \text{using} \quad \text{eigenvecs}\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) \rightarrow \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (a-b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ solve}, y \rightarrow -x \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (a+b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ solve}, y \rightarrow x \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Completeness relations:

$$\text{Hard} \cdot \text{Hard}^T + \text{Soft} \cdot \text{Soft}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{Black} \cdot \text{Black}^T + \text{White} \cdot \text{White}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Sw} \cdot \left(\overline{\text{Sw}}\right)^T + \text{So} \cdot \left(\overline{\text{So}}\right)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{P} \cdot \text{P}^T + \text{F} \cdot \text{F}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Operators from eigenfunctions:

$$\text{Hard} \cdot \text{Hard}^T - \text{Soft} \cdot \text{Soft}^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{Black} \cdot \text{Black}^T - \text{White} \cdot \text{White}^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Sw} \cdot \left(\overline{\text{Sw}}\right)^T - \text{So} \cdot \left(\overline{\text{So}}\right)^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{P} \cdot \text{P}^T - \text{F} \cdot \text{F}^T = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

The hardness, color, taste and odor operators are Hermitian and unitary:

$$\left(\overline{\text{Hardness}}\right)^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Hardness} \cdot \text{Hardness} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left(\overline{\text{Color}}\right)^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Color} \cdot \text{Color} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\overline{\text{Taste}}\right)^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{Taste} \cdot \text{Taste} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left(\overline{\text{Odor}}\right)^T = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix} \quad \text{Odor} \cdot \text{Odor} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Appendix B: Additional Computational Details

In order to explain the perfect anti-correlation predicted by quantum mechanics when the same type of measurement is made on both quons, the local realist makes the following state assignments. Remember that these states are not legitimate according to quantum theory because hardness, color, taste and odor are incompatible observables.

QuonA	QuonB
HBSwP	SWSoF
HBSwF	SWSoP
HBSoP	SWSwF
HBSoF	SWSwP
HWSwP	SBSoF
HWSwF	SBSoP
HWSoP	SBSwF
HWSoF	SBSwP

Property	Eigenvalue
H	1
S	−1
B	1
W	−1
Sw	1
So	−1
P	1
F	−1

It is easy to show that these state assignments are in agreement with the following quantum mechanical calculations.

$\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Hardness}) \cdot \Psi = -1$ $\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Color}) \cdot \Psi = -1$ $\Psi^T \cdot \text{kronecker}(\text{Taste}, \text{Taste}) \cdot \Psi = -1$

$\Psi^T \cdot \text{kronecker}(\text{Odor}, \text{Odor}) \cdot \Psi = -1$ $\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Color}) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Taste}) \cdot \Psi = 0$

$\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Taste}) \cdot \Psi = 0$ $\Psi^T \cdot \text{kronecker}(\text{Taste}, \text{Odor}) \cdot \Psi = 0$

However, the state assignments are not consistent with the following quantum calculations.

$\Psi^T \cdot \text{kronecker}(\text{Hardness}, \text{Odor}) \cdot \Psi = -0.707$ $\Psi^T \cdot \text{kronecker}(\text{Color}, \text{Odor}) \cdot \Psi = -0.707$

The last two rows of the following table compare the local realist and quantum mechanical predictions, showing the disagreement for the hardness/odor and color/odor joint measurements.

QuonA	QuonB	HH	CC	TT	OO	HC	HT	HO	CT	CO	TO
HBSwP	SWSoF	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
HBSwF	SWSoP	−1	−1	−1	−1	−1	−1	1	−1	1	1
HBSoP	SWSwF	−1	−1	−1	−1	−1	1	−1	1	−1	1
HBSoF	SWSwP	−1	−1	−1	−1	−1	1	1	1	1	−1
HWSwP	SBSoF	−1	−1	−1	−1	1	−1	−1	1	1	−1
HWSwF	SBSoP	−1	−1	−1	−1	1	−1	1	1	−1	1
HWSoP	SBSwF	−1	−1	−1	−1	1	1	−1	−1	1	1
HWSoF	SBSwP	−1	−1	−1	−1	1	1	1	−1	−1	−1
Expectation	Value	−1	−1	−1	−1	0	0	0	0	0	0
Quantum	Result	−1	−1	−1	−1	0	0	−0.707	0	−0.707	0