Aspects of Dirac's Relativistic Matrix Mechanics

Frank Rioux

This tutorial provides a brief summary of the last chapter of C. W. Sherwin's excellent *Introduction to Quantum Mechanics* which deals with relativistic quantum mechanics.

The relativistic equation for the energy of a free particle has positive and negative roots, where the positive root signifies the energy of a particle and the negative root the energy of its antiparticle. This interpretation was confirmed experimentally with the discovery of the anti-electron (positron) in 1932 by Anderson.

$$E = \pm c\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2c^2} \qquad (1)$$

Dirac converted this to a soluble quantum mechanical operator by first writing the argument of the square root as a perfect square in order to get rid of the troubling radical operator which defied physical interpretation. In a second step he replaced energy and momentum with their differential operators, E = -(h/2πi)d/dt and p_q = (h/2πi)d/dq, from non-relativistic quantum mechanics.

$$p_x^2 + p_y^2 + p_z^2 + m^2 \cdot c^2 = \left(\alpha_x \cdot p_x + \alpha_y \cdot p_y + \alpha_z \cdot p_z + \beta \cdot m \cdot c\right)^2$$

For this mathematical maneuver to be valid the following conditions must hold: $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

$$\alpha_x \cdot \alpha_y + \alpha_y \cdot \alpha_x = 0 \qquad \alpha_x \cdot \alpha_z + \alpha_z \cdot \alpha_x = 0 \qquad \alpha_x \cdot \beta + \beta \cdot \alpha_x = 0 \qquad \alpha_y \cdot \alpha_z + \alpha_z \cdot \alpha_y = 0$$

$$\alpha_y \cdot \beta + \beta \cdot \alpha_y = 0 \qquad \alpha_z \cdot \beta + \beta \cdot \alpha_z = 0 \qquad p_x \cdot p_y = p_y \cdot p_x \qquad p_x \cdot p_z = p_z \cdot p_x \qquad p_y \cdot p_z = p_z \cdot p_y$$

In other words, the αs and β must anticommute while the momentum operators as used above must commute. From the non-relativistic formulation of quantum mechanics it was already clear that the momentum operator pairs above did commute. In formulating a relativistic quantum mechanics, Dirac assumed the validity of the various multiplicative and differential operators of non-relativistic quantum mechanics for observable properties like energy, position and momentum.

Being cognizant of Heisenberg's matrix approach to non-relativistic quantum mechanics, Dirac realized the restrictions above regarding the αs and β could be satisfied by the following 4x4 matrices.

$$\alpha_x := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_y := \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_z := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \beta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

First we show that $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I$ where I is the identity operator.

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_x \cdot \alpha_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \alpha_y \cdot \alpha_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \alpha_z \cdot \alpha_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta \cdot \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we show that the αs and β anticommute:

$$\alpha_x \cdot \alpha_y + \alpha_y \cdot \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_x \cdot \alpha_z + \alpha_z \cdot \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_x \cdot \beta + \beta \cdot \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_y \cdot \alpha_z + \alpha_z \cdot \alpha_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_y \cdot \beta + \beta \cdot \alpha_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_z \cdot \beta + \beta \cdot \alpha_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is now possible to write Dirac's relativistic energy equation as follows:

$$E = \pm c\left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc\right) \tag{2}$$

Before proceeding to the next step, the substitution of the differential operators for energy and momentum, it is instructive to look at the right side of the above equation which is a 4x4 Dirac relativistic energy operator. Of course, the left side is a 4x4 matrix with E on the diagonal and zeros everywhere else.

$$-c \cdot \left(\alpha_x \cdot p_x + \alpha_y \cdot p_y + \alpha_z \cdot p_z + \beta \cdot m \cdot c\right) \rightarrow \begin{bmatrix} -c^2 \cdot m & 0 & -c \cdot p_z & -c \cdot \left(p_x - p_y \cdot i\right) \\ 0 & -c^2 \cdot m & -c \cdot \left(p_x + p_y \cdot i\right) & c \cdot p_z \\ -c \cdot p_z & -c \cdot \left(p_x - p_y \cdot i\right) & c^2 \cdot m & 0 \\ -c \cdot \left(p_x + p_y \cdot i\right) & c \cdot p_z & 0 & c^2 \cdot m \end{bmatrix} \tag{3}$$

Substituting the traditional operators for energy and momentum yields,

$$-\frac{\hbar}{i}\frac{\partial \Psi}{\partial t} = -\left[\frac{c\hbar}{i}\left(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z}\right) + \beta mc^2\right]\Psi \tag{4}$$

Assuming the separability of the space and time coordinates [$\Psi(x,y,z,t) = \psi(x,y,z)\phi(t)$], this four dimensional differential equation is decoupled in to two differential equations. The time-dependent equation is easily solved and has the following solution.

$$\varphi(t) = e^{-i\frac{Et}{\hbar}} \tag{5}$$

The space part of the differential equation has the following form, with the relativistic Hamiltonian operating on the wave function.

$$-\left[\frac{c\hbar}{i}\left(\alpha_x\frac{\partial}{\partial x}+\alpha_y\frac{\partial}{\partial y}+\alpha_z\frac{\partial}{\partial z}\right)+\beta mc^2\right]\psi = E\psi \tag{6}$$

As demonstrated above (eqn 3) the relativistic energy operator is a 4x4 matrix. Therefore, the wave function must be a four-component vector.

At this point Sherwin turns to the example of the free particle in the x-direction (see pages 292-295). He assumes that the solution has the form of a plane wave. However, as shown below substitution of the deBroglie equation in the plane wave equation yields the momentum eigenfunction in coordinate space.

$$\exp\left(i\frac{2\pi}{\lambda}x\right)\xrightarrow{\lambda = h/p}\exp\left(i\frac{px}{\hbar}\right)$$

This means that this problem is extremely easy to solve in momentum space where the momentum operator is multiplicative. The calculation of the energy eigenvalues is straight forward using Mathcad's ***eigenvals*** command. We simply ask for the eigenvalues of the relativistic energy operator as shown below.

$$\text{eigenvals}\left[-c\cdot\left(\alpha_x\cdot p_x+\beta\cdot m\cdot c\right)\right]\rightarrow\begin{pmatrix} c\cdot\sqrt{c^2\cdot m^2+p_x^{\,2}} \\ c\cdot\sqrt{c^2\cdot m^2+p_x^{\,2}} \\ -c\cdot\sqrt{c^2\cdot m^2+p_x^{\,2}} \\ -c\cdot\sqrt{c^2\cdot m^2+p_x^{\,2}} \end{pmatrix}$$

Calculation of the (unnormalized) eigenvectors is equally easy.

$$\text{eigenvecs}\left[-c\cdot\left(\alpha_x\cdot p_x+\beta\cdot m\cdot c\right)\right] = \begin{pmatrix} \frac{W+m\cdot c^2}{p_x\cdot c} & 0 & 0 & \frac{-W+m\cdot c^2}{p_x\cdot c} \\ 0 & \frac{W+m\cdot c^2}{p_x\cdot c} & \frac{-W+m\cdot c^2}{p_x\cdot c} & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad W = \sqrt{p_x^{\,2}\cdot c^2+m^2\cdot c^4}$$