The Dirac Delta Function

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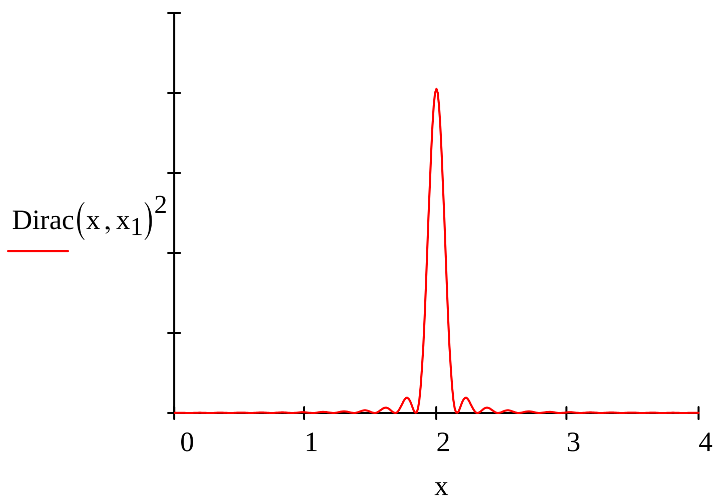
The Dirac delta function expressed in Dirac notation is: $\Delta(x - x_1) = \langle x | x_1 \rangle$. The $\langle x | x_1 \rangle$ bracket is evaluated using the momentum completeness condition. See the Mathematical Appendix for definitions of the required Dirac brackets and other mathematical tools used in the analysis that follows.

$$\langle x | x_1 \rangle = \int_{-\infty}^{\infty} \langle x | p \rangle \langle p | x_1 \rangle dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ipx)\exp(-ipx_1) dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[ip(x - x_1)] dp$$

Evaluation of this integral over a finite range of momentum values shows that the delta function is small except in the immediate neighborhood of x_1. Integrating from -20 to 20 to reduce computational time shows that $\langle x | x_1{=}2 \rangle$ is small except in the area x = 2.

$$x_1 := 2 \qquad x := 0, .01 .. 4 \qquad \mathrm{Dirac}(x, x_1) := \frac{1}{2 \cdot \pi} \cdot \int_{-20}^{20} \exp[i \cdot p \cdot (x - x_1)] \, dp$$



The Fourier transform of the Dirac delta function into the momentum representation yields the following result.

$$\int_{-\infty}^{\infty} \langle p | x \rangle \langle x | x_1 \rangle dx = \frac{1}{\sqrt{2\pi}} \exp(-ipx_1) = \langle p | x_1 \rangle$$

The normalization constant is omitted for clarity of expression and the previous value of x_1 is cleared to allow symbolic calculation.

$$x_1 := x_1 \qquad \int_{-\infty}^{\infty} \exp(-i \cdot p \cdot x) \cdot \Delta(x - x_1) \, dx \text{ simplify } \rightarrow e^{-p \cdot x_1 \cdot i}$$

Mathematical Appendix

The position and momentum completeness conditions: $\int |x\rangle\langle x| dx = 1 \qquad \int |p\rangle\langle p| dp = 1$

The momentum eigenstate in the coordinate representation: $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} \exp(ipx)$

The position eigenstate in the momentum representation: $\langle p|x\rangle = \frac{1}{\sqrt{2\pi}} \exp(-ipx)$