Fourier Synthesis

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The purpose of this tutorial is to use Dirac notation to examine Fourier synthesis. The first step is to write the function symbolically in Dirac notation.

$$f(x) = \langle x | f \rangle \qquad (1)$$

Select an orthonormal basis set, |n>, for which the completeness relation holds.

$$\sum_n |n\rangle\langle n| = 1 \qquad (2)$$

Expand |f> in terms of |n> by inserting equation (2) into the right side of equation (1). In other words write f(x) as a weighted (<n|f>) superposition using the <x|n> basis set (the |n> basis set expressed in the coordinate representation).

$$f(x) = \sum_n \langle x|n\rangle\langle n|f\rangle \qquad (3)$$

Evaluate the Fourier coefficient, <n|f>, using the continuous completeness relation in coordinate space.

$$\int |x'\rangle\langle x'| dx' = 1 \qquad (4)$$

Equation (3) becomes,

$$f(x) = \sum_n \langle x|n\rangle \int \langle n|x'\rangle\langle x'|f\rangle dx' \qquad (5)$$

Now select a function

$$\langle x'|f\rangle = x'^3(1 - x') \qquad (6)$$

over the interval (0,1). Choose the following orthonormal basis set over the same interval.

$$\langle x|n\rangle = \sqrt{2}\sin(n\pi x) \qquad (7)$$

Substitution of equations (6) and (7) into (5) yields

$$f(x) = \sum_n \sqrt{2}\sin(n\pi x)\int_0^1 \sqrt{2}\sin(n\pi x')x'^3(1 - x')dx' \qquad (8)$$

The Fourier synthesis and the original function are shown for n = 2, 4, and 10 in the figure below.

$$x := 0, .025 .. 1.0 \qquad f(x,n) := \sum_{i=1}^{n} \left[\sqrt{2} \cdot \sin(i \cdot \pi \cdot x) \cdot \int_0^1 \sqrt{2} \cdot \sin(i \cdot \pi \cdot x') \cdot x'^3 \cdot (1 - x')\, dx' \right]$$