Variation Calculation for a Particle in a Gravitational Field in Momentum Space

The following problem deals with a particle of unit mass in a gravitational field with acceleration due to gravity equal to 1.

Energy operator for particles near Earth's surface: $-\frac{1}{2\cdot\mu}\cdot\frac{d^2}{dz^2}\blacksquare + z\cdot\blacksquare$

Trial wave function: $\Psi(\alpha, z) := 2\cdot\alpha^{\frac{3}{2}}\cdot z\cdot\exp(-\alpha\cdot z)$

Fourier the position wave function into momentum space:

$$\Phi(\alpha, p) := \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_0^{\infty}\exp(-i\cdot p\cdot z)\cdot\Psi(\alpha, z)\,dz \;\Big|\; \begin{array}{l}\text{assume}, \alpha > 0\\ \text{simplify}\end{array} \rightarrow \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\cdot\frac{\alpha^{\frac{3}{2}}}{(i\cdot p + \alpha)^2}$$

Demonstrate that the momentum wave function is normalized.

$$\int_{-\infty}^{\infty}\overline{\Phi(\alpha, p)}\cdot\Phi(\alpha, p)\,dp \text{ assume}, \alpha > 0 \;\rightarrow 1$$

Energy operator in momentum space: $\frac{p^2}{2}\cdot\blacksquare + i\cdot\frac{d}{dp}\blacksquare$

Evaluate the variational expression for the energy:

$$E(\alpha) := \int_{-\infty}^{\infty}\overline{\Phi(\alpha, p)}\cdot\frac{p^2}{2}\cdot\Phi(\alpha, p)\,dp \;\ldots \qquad \text{assume}, \alpha > 0 \;\rightarrow \frac{1}{2}\cdot\alpha^2 + \frac{3}{2\cdot\alpha}$$
$$+ \int_{-\infty}^{\infty}\overline{\Phi(\alpha, p)}\cdot i\cdot\left(\frac{d}{dp}\Phi(\alpha, p)\right)dp$$

Minimize energy with respect to variational parameter α:

$\alpha := 1$ $\qquad \alpha := \text{Minimize}(E, \alpha)$ $\qquad \alpha = 1.145$ $\qquad E(\alpha) = 1.966$

This momentum space result is in exact agreement with the coordinate-space result. The exact value for the energy is 1.856.

$$\frac{E(\alpha) - 1.856}{1.856} = 5.9\%$$